Asymptotic Analysis of a Model in Dendritic Solidification Subjected to Buoyancy-driven Convection in the Far Field

Tat Leung Yee
Department of Mathematics and Information Technology,
The Education University of Hong Kong,
Tai Po, New Territories, Hong Kong.

Abstract
We first recall the mathematical formulation of the model of a single needle dendrite solidifying in an under-cooled environment. The solidification process is supposed to be affected by the buoyancy-driven convection in the far field. We next recall the results on the asymptotic formulation for the mathematical model in the limit of the small gravity parameter and we derived the leading-order asymptotic expansion solutions of the flow field and the temperature field in the near field as well as in the far field. The inner and outer solutions are supposed to be matched in an intermediate region. In this paper, the first-order asymptotic governing equations and the corresponding boundary conditions have also been formulated.

AMS subject classification: 30E15, 35Q30, 35Q79.
Keywords: Asymptotic analysis, dendritic solidification, buoyancy effect.

1. Introduction
It has been a long history to investigate the dendritic solidification in an under-cooled melt experimentally [1]–[2]. The great interest of the study of dendritic growth of a needle crystal by scientists has raised the curiosity of mathematicians [3]–[4]. A rigorous mathematical formulation of the dynamical system has been established and Xu [5]–[8] published a series of papers focusing the mathematical methods in the problems such as the selection of tip velocity and the origin of pattern formation in dendritic growth.

In the literature, the first important analytic result was the Ivantsov’s solution. He considered the steady state dendritic growth problem with zero surface tension. The solution can not determine the top radius and tip velocity separately, however, the explicit
solution provides some insights of the problem with nonzero surface tension. Besides, some researchers made some extensive and detailed experiments and examined the pattern formation for dendritic growth. The selection condition of the tip velocity has been found in the isotropic surface tension case.

A new theory called the Interfacial Wave Theory of solidification has been raised by Xu [5]–[8], which gives very good agreement with experimental observations. Later on, Xu introduced the method of asymptotic expansions in dendritic growth problems and has found a uniformly valid asymptotic expansion solution, for large Prandtl number. The dendritic growth under the convection motion induced by an oscillating external source has been recently investigated [9]–[11]. We also studied the dendritic growth problem due to buoyancy [12], in which we succeeded to find the leading-order asymptotic solution which is valid in the tip region of the dendritic growth. In this paper, we investigate the dendritic crystal growth problem under the effect of buoyancy-driven convection in the far field of the dendritic growth. We shall look for the leading-order asymptotic expansion solutions of the temperature field and the interface shape function in the far field by assuming the gravity parameter is small.

In Sect. 2, the mathematical formulation of the dendritic growth problem will be presented in detail. The corresponding dimensionless governing equations and dimensionless boundary conditions will be given. In Sect. 3, we shall derive the leading-order asymptotic expansion solutions as well as presenting the first-order approximations of the partial differential system by assuming the asymptotic expansions of the flow field and the temperature field far away from the tip region of the dendritic growth.

2. Mathematical formulation

The crystal growth of needle dendrite during solidification has been intensively studied by physicists and material scientists for a long time. One important foundation work is to formulate the problem into a feasible mathematical model. The general mathematical formulation of the needle dendritic growth is introduced in the following. The major physical transport process in a pure melt is heat conduction. The under-cooling temperature of the melt is $T_\infty$. The melt is considered as an incompressible Newtonian fluid and is assumed to be infinite in extent. The dendrite is supposed to grow under the convection due to buoyancy effect. The single needle dendrite which is growing into an under-cooled pure melt along the $Z$-axis direction with a constant tip velocity $V$, with zero surface tension on the interface between liquid and solid states. Assume that the thermal diffusivity $\kappa_T$ and the heat capacity $c_p$ of the liquid state are the same as those of the solid state, the mass density of liquid state is $\rho$ and the mass density of solid state is $\rho_s$. Let $U$ be the absolute velocity field of the fluid and $T$ be the temperature field in the liquid melt.

The governing equations of the dynamical heat transfer model consisting of the Navier-Stokes equations with Boussinesq approximation applied, the mass continuity
equation, and the heat conduction equations for the liquid and solid states are as follows:

\[ \frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times U) = v \nabla^2 \Omega + \nabla \times [\beta (T - T_{\infty}) g e_Z], \quad \Omega = \nabla \times U, \]

\[ \nabla \cdot U = 0, \quad \frac{\partial T}{\partial t} + U \cdot \nabla T = \kappa T \nabla^2 T, \quad \frac{\partial T_s}{\partial t} = \kappa T \nabla^2 T_s, \]

where \( T \) and \( T_s \) denote the temperature fields, \( U \) denotes the absolute velocity field of the fluid motion, \( t \) denotes the time, \( e_Z \) the unit vector along the \( Z \)-axis, and \( v \) is the kinematic viscosity, \( \beta \) is the thermal expansion coefficient, \( P \) is the reduce pressure, \( g \) is the acceleration of gravity.

The boundary conditions are given by the continuity condition of temperature, the Gibbs-Thomson condition (the temperature at the interface depends on the local interface curvature), the enthalpy conservation condition, the mass conservation condition, the continuity condition of tangential component of velocity. Here we skip the descriptions of all the dimensionless boundary conditions.

In order to simplify the above partial differential system, the first step will be the introduction of the change of dimensionless variables. In fact, the standard coordinate system being used will be the parabolic cylindrical coordinate system. We denote the coordinates \((\xi, \eta, \theta)\) as moving with the constant velocity \( V \) such that \( r = \eta_0^2 \xi \eta \), \( 2z = \eta_0^2 (\xi^2 - \eta^2) \). In the above, the parameter \( \eta_0^2 \) could be determined by normalizing the location of the tip of the dendrite at the location \( \eta = 1 \) and that normalized value of \( \eta \) will characterize the interface shape function. In this paper, we attempt to derive the leading-order asymptotic expansion solutions of the temperature field and the interface shape function in the far field. We shall consider the dendritic growth with the presence of the effect of the buoyancy-driven convection. We assume that gravity is taken to be negligible, so that the dimensionless gravity parameter will be taken to approaching zero. In the following we shall skip the non-dimensionalized governing equations and the boundary conditions in terms of the paraboloidal coordinates. We express the dimensionless governing system in the following. Note also that the following are two important dimensionless parameters:

\[ \text{Pr} = \frac{v}{\kappa T} \quad \text{and} \quad \text{Gr} = \frac{g \beta (T_{M0} - T_{\infty}) \kappa T}{V^3}. \]

The first parameter is the Prandtl number whereas the second parameter is the dimensionless gravitational parameter.

Consider the single dendritic growth under the effect of buoyancy-driven convection. The dendrite is subject to the small gravity which means the gravity parameter \( \text{Gr} \) is negligible. We also assume that the surface tension is negligible. Therefore we only need to consider the temperature field in liquid melt. The governing equations are:

\[ D_\xi \Psi = -\left( \xi^2 + \eta^2 \right) \zeta, \quad (1) \]

\[ \text{Pr} \ D_\zeta = \eta_0^2 D_\xi \zeta + \frac{2 \zeta}{\eta_0^2 \xi^2 \eta^2} D_\xi \Psi - \frac{1}{\eta_0^2 \xi \eta} \left( \Psi_{\xi \eta} - \Psi_{\eta \xi} \right) - \frac{\text{Gr} \ \xi \eta}{T_{\infty}} D_\eta T, \quad (2) \]
\[ D_2^2 T = \eta_0^2 D_1^2 T + \frac{1}{\eta_0^2 \xi \eta} \left( \Psi_{\eta} T_{\xi} - \Psi_{\xi} T_{\eta} \right), \]  

(3)

where the differential operators are defined as

\[
D_2^- = \left\{ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{\eta} \frac{\partial}{\partial \eta} \right\}, \quad D_2^+ = \left\{ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{1}{\eta} \frac{\partial}{\partial \eta} \right\},
\]

\[
D_1^- = \left\{ \xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right\}, \quad D_1^+ = \left\{ \eta \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \eta} \right\}.
\]

We attempt to study the far field asymptotic expansion solutions of the flow field \( \Psi \) and the temperature field \( T \) under the assumption that the gravitational acceleration is negligible. The gravity parameter \( \text{Gr} \) is assumed to be small and this is regarded as the small parameter in the asymptotic analysis. The mathematical formulation of the dendritic growth will allow us to derive the asymptotic expansion solutions in the orders of \( \text{Gr} \). The leading-order asymptotic expansion solutions will be the zero-th order solutions. Note also that the entire transport field can be considered as near the tip and far away from the tip. We usually describe the asymptotic expansion solutions in the near field (inner solutions) and in the far field (outer solutions), respectively. The inner solutions and the outer solutions should be matched in an intermediate region and the matched solution is regarded as the globally valid asymptotic solution in the whole physical field.

3. Asymptotic analysis in the far field \( \eta \gg \xi \)

In [12], we have successfully obtained the zero-order inner expansion solution. We recall the solution as follows,

\[
\begin{align*}
\hat{\zeta}_0(\xi, \tilde{\eta}) & = a_0 \xi^2 \tilde{\eta}^2 + c_0, \\
\hat{\psi}_0(\xi, \tilde{\eta}) & = \frac{\xi^2}{8} \left[ a_0 \xi^2 (\tilde{\eta}^4 - 2 \tilde{\eta}^2 + 1) + 2c_0 \left( 2 \tilde{\eta}^2 \ln \tilde{\eta} - \tilde{\eta}^2 + 1 \right) \right], \\
\hat{T}_0(\tilde{\eta}) & = -\eta_0^2 \ln \tilde{\eta}.
\end{align*}
\]

(4)

which contains arbitrary constants \( a_0 \) and \( c_0 \). In fact, the two constants could be determined in principle by matching the inner solution with the outer solution. Now we deduce the zero-order outer solution in details. We note that the zero-order outer solution is by definition valid in the far field. It could be seen that solving the system in the far field is actually a singular perturbation problem. Thus we intend to use the multiple-value expansion method in the following. By applying the method, we shall use the new outer variables \((\xi, \tilde{\eta}, \eta_+)\) where \( \tilde{\eta} \) is a slow variable and \( \eta_+ \) is a fast variable. The variables, in this case, are defined as

\[
\begin{align*}
\tilde{\eta} & = \text{Gr}^{-\frac{1}{2}} \eta, \\
\eta_+ & = \frac{1}{\text{Gr}^{\frac{1}{2}}} \int_{0}^{\tilde{\eta}} k(\eta_1, \text{Gr}) \, d\eta_1.
\end{align*}
\]

(5)
where $k$ is an undetermined function of the slow variable $\tilde{\eta}$. We consider the exact solution as a function of $(\xi, \tilde{\eta}, \eta_+)$ where the fast variable and the slow variable are formally treated as independent variables. Thus, we write the solution in the following multiple-value expansion form in terms of the small parameter $\text{Gr}$:

$$
\Psi(\xi, \eta, \text{Gr}) = \tilde{\Psi}(\xi, \tilde{\eta}, \eta_+, \text{Gr}),
$$

$$
\zeta(\xi, \eta, \text{Gr}) = \tilde{\zeta}(\xi, \tilde{\eta}, \eta_+, \text{Gr}),
$$

$$
T(\xi, \eta, \text{Gr}) = \tilde{T}(\xi, \tilde{\eta}, \eta_+, \text{Gr}).
$$

(6)

Now we need to transform the original system into the form with the above multiple variables. This can be done by replacing all the derivatives in (1)–(3) by using the following transformations:

$$
\frac{\partial}{\partial \eta} \Rightarrow k \frac{\partial}{\partial \eta} + \text{Gr}^2 \frac{\partial}{\partial \tilde{\eta}},
$$

$$
\frac{\partial^2}{\partial \eta^2} \Rightarrow k^2 \frac{\partial^2}{\partial \eta_+^2} + 2k \frac{\partial^2}{\partial \eta_+ \partial \eta} + k' \frac{\partial}{\partial \eta_+} + \text{Gr} \frac{\partial^2}{\partial \eta^2}.
$$

(7)

The multiple-variable form of the original governing system is then obtained as follows:

$$
\left\{ \begin{array}{c}
k^2 \frac{\partial^2}{\partial \eta_+^2} + \text{Gr} \left( 2k \frac{\partial^2}{\partial \eta_+ \partial \eta} + k' \frac{\partial}{\partial \eta_+} - k \frac{\partial}{\tilde{\eta} \partial \eta_+} \right) + \text{Gr}^2 \left( \frac{\partial^2}{\partial \eta^2} - \frac{1}{\tilde{\eta}} \frac{\partial}{\partial \tilde{\eta}} \right) \end{array} \right\} \tilde{\Psi}
$$

$$
= -\tilde{\eta}^2 \tilde{\xi} - \text{Gr} \left( \frac{\partial^2 \tilde{\Psi}}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial \tilde{\Psi}}{\partial \xi} \right) - \text{Gr} \xi \tilde{\eta} \cdot \tilde{\xi},
$$

(8)

$$
\text{Pr} \left\{ \begin{array}{c}
k^2 \frac{\partial^2}{\partial \eta_+^2} + \text{Gr} \left( 2k \frac{\partial^2}{\partial \eta_+ \partial \eta} + k' \frac{\partial}{\partial \eta_+} - k \frac{\partial}{\tilde{\eta} \partial \eta_+} \right) + \text{Gr}^2 \left( \frac{\partial^2}{\partial \eta^2} - \frac{1}{\tilde{\eta}} \frac{\partial}{\partial \tilde{\eta}} \right) \end{array} \right\} \tilde{\zeta}
$$

$$
= -\eta_0 k \frac{\partial \tilde{\zeta}}{\partial \eta_+} - \text{Gr} \ \text{Pr} \left( \frac{\partial^2 \tilde{\zeta}}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial \tilde{\zeta}}{\partial \xi} \right) + \text{Gr} \eta_0 \left( \frac{\partial \tilde{\zeta}}{\partial \xi} - \tilde{\eta} \frac{\partial \tilde{\zeta}}{\partial \tilde{\eta}} \right)
$$

$$
+ \text{Gr} \frac{2 \tilde{\xi}}{\eta_0 \xi^2 \tilde{\eta}^2} \left[ -k \tilde{\eta} \frac{\partial \tilde{\Psi}}{\partial \eta_+} + \text{Gr} \left( \frac{\partial \tilde{\Psi}}{\partial \xi} - \tilde{\eta} \frac{\partial \tilde{\Psi}}{\partial \tilde{\eta}} \right) \right]
$$

$$
- \frac{\text{Gr}}{\eta_0 \tilde{\xi} \tilde{\eta}} \left[ k \frac{\partial \tilde{\Psi}}{\partial \xi} \frac{\partial \tilde{\zeta}}{\partial \eta_+} - k \frac{\partial \tilde{\Psi}}{\partial \eta_+} \frac{\partial \tilde{\xi}}{\partial \xi} + \text{Gr} \left( \frac{\partial \tilde{\Psi}}{\partial \xi} \frac{\partial \tilde{\zeta}}{\partial \eta_+} - \frac{\partial \tilde{\Psi}}{\partial \eta_+} \frac{\partial \tilde{\xi}}{\partial \xi} \right) \right]
$$

$$
- \frac{\text{Gr} \xi \tilde{\eta}}{T_\infty} \left( \frac{\partial \tilde{T}}{\partial \xi} + k \xi \frac{\partial \tilde{T}}{\partial \eta_+} + \text{Gr} \xi \frac{\partial \tilde{T}}{\partial \tilde{\eta}} \right).
$$

(9)
\[
\left\{ \frac{k^2}{\eta_+^2} \frac{\partial^2}{\partial \eta_+^2} + \operatorname{Gr} \left( 2k \frac{\partial^2}{\partial \eta_+ \partial \tilde{\eta}} + k' \frac{\partial}{\partial \eta_+} + k \frac{\partial}{\partial \tilde{\eta}} \frac{\partial}{\partial \eta_+} \right) + \operatorname{Gr}^2 \left( \frac{\partial^2}{\partial \eta_+^2} + \frac{1}{\tilde{\eta}} \frac{\partial}{\partial \tilde{\eta}} \frac{\partial}{\partial \eta_+} \right) \right\} \tilde{T} = -\eta_0^2 k \eta \frac{\partial \tilde{T}}{\partial \eta_+} - \operatorname{Gr} \left( \frac{\partial^2 \tilde{T}}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \tilde{T}}{\partial \xi} \right) + \operatorname{Gr} \eta_0^2 \left( \frac{\xi}{\tilde{\eta}} \frac{\partial \tilde{T}}{\partial \xi} - \frac{\tilde{\eta}}{\tilde{\eta}} \frac{\partial \tilde{T}}{\partial \eta_+} \right), \tag{10} \right. \\
\left. + \operatorname{Gr} \frac{\frac{\partial \Psi}{\partial \eta_+} \frac{\partial \tilde{T}}{\partial \xi} - \frac{k}{\eta_0 \xi} \frac{\partial \tilde{T}}{\partial \eta_+} + \operatorname{Gr} \frac{\partial \Psi}{\partial \tilde{\eta}} \frac{\partial \tilde{T}}{\partial \xi} - \frac{\tilde{\eta}}{\tilde{\eta}} \frac{\partial \tilde{T}}{\partial \eta_+} \right), \right. \\
\text{with the far field boundary conditions at } \tilde{\eta} \to \infty \text{ and } \eta_+ \to \infty, \\
\tilde{\Psi} \to 0, \tag{11} \\
\tilde{\zeta} \to 0, \tag{12} \\
\tilde{T} \to T_\infty. \tag{13} \\
\right.
\]

Finding the exact solution for nonlinear PDE system (8)–(13) in the far field is supposed to be very difficult. Instead, we attempt to find its asymptotic expansion solution in the limit \( \operatorname{Gr} \to 0 \). Just like what we have done for handling inner region asymptotic expansions, we can make the following multiple-value expansion for the perturbed state:

\[
\begin{align*}
\tilde{\Psi} &= \operatorname{Gr}^\beta \left\{ \Psi_0(\xi, \tilde{\eta}, \eta_+) + \operatorname{Gr} \Psi_1(\xi, \tilde{\eta}, \eta_+) + \cdots \right\}, \\
\tilde{\zeta} &= \operatorname{Gr}^\beta \left\{ \zeta_0(\xi, \tilde{\eta}, \eta_+) + \operatorname{Gr} \zeta_1(\xi, \tilde{\eta}, \eta_+) + \cdots \right\}, \\
\tilde{T} &= T_\infty + \operatorname{Gr} \left\{ \tilde{T}_0(\xi, \tilde{\eta}, \eta_+) + \operatorname{Gr} \tilde{T}_1(\xi, \tilde{\eta}, \eta_+) + \cdots \right\}, \tag{14} \\
k &= k_0(\tilde{\eta}) + \operatorname{Gr} k_1(\tilde{\eta}) + \cdots, 
\end{align*}
\]

where \( T_\infty \) is the under-cooling temperature, which is often regarded as the control parameter in experiment. As we stated, the parameter \( \eta_0^2 \) is determined by the under-cooling \( T_\infty \), namely

\[
T_\infty = -\eta_0^2 \frac{1}{\eta_0^2} \int_1^\infty e^{-\frac{1}{2} \eta_0^2 \eta_1^2} \frac{d\eta_1}{\eta_1}. \tag{15} 
\]

By substituting (14) into the system (8)–(13), one can derive each order of approximation in the outer region. From (14), the exponent \( \beta \) of \( \operatorname{Gr} \) could be determined. It is seen that in order to balance the leading order terms of the Eq.(9), one must set \( \beta = 2 \). Hence, as the leading order approximation, one could derive the following system of equations for zero-order outer solution:

\[
\begin{align*}
k_0^2 \frac{\partial^2 \tilde{\Psi}_0}{\partial \eta_+^2} &= -\tilde{\eta}^2 \zeta_0, \tag{16} \\
\Pr k_0^2 \frac{\partial^2 \zeta_0}{\partial \eta_+^2} &= -\eta_0^2 k_0 \frac{\partial \tilde{\zeta}_0}{\partial \eta_+} - \frac{\xi \tilde{\eta}^2}{T_\infty} \frac{\partial \tilde{T}_0}{\partial \xi} - k_0 \frac{\tilde{\eta}^2}{T_\infty} \frac{\partial \tilde{T}_0}{\partial \eta_+}, \tag{17} 
\end{align*}
\]
Asymptotic Analysis of a Model in Dendritic Solidification

\[ k_0 \frac{\partial^2 \tilde{T}_0}{\partial \eta_+^2} = -\eta_0^2 \frac{\partial \tilde{T}_0}{\partial \eta_+}, \]  

(18)

with boundary conditions at \( \tilde{\eta} \to \infty \) and \( \eta_+ \to \infty \),

\[ \tilde{\Psi}_0 \to 0, \]  

(19)

\[ \tilde{\zeta}_0 \to 0, \]  

(20)

\[ \tilde{T}_0 \to 0. \]  

(21)

The above system (16)–(21) is simple enough to be explicitly solved. In order to solve the system, the temperature \( \tilde{T}_0 \) should be obtained firstly and then \( \tilde{\zeta}_0 \) and finally the flow field \( \tilde{\Psi} \). In fact, we have derived that the zero-order outer solution and the result is given:

\[
\begin{align*}
\tilde{\zeta}_0 &= \xi^2 \left[ \frac{k_0 B(\tilde{\eta})}{\eta_0^2 (Pr - 1) T_\infty} e^{-\frac{\eta_0^2 \tilde{\eta} \eta_+}{k_0}} + C(\tilde{\eta}) e^{-\frac{\eta_0^2 \tilde{\eta} \eta_+}{k_0}} \right], \\
\tilde{\Psi}_0 &= -\xi^2 \left[ \frac{k_0 B(\tilde{\eta})}{\eta_0^2 (Pr - 1) T_\infty} e^{-\frac{\eta_0^2 \tilde{\eta} \eta_+}{k_0}} + \frac{Pr^2 C(\tilde{\eta})}{\eta_0^4} e^{-\frac{\eta_0^2 \tilde{\eta} \eta_+}{k_0}} + E(\tilde{\eta}) \eta_+ + F(\tilde{\eta}) \right], \\
\tilde{T}_0 &= A(\tilde{\eta}) + B(\tilde{\eta}) e^{-\frac{\eta_0^2 \tilde{\eta} \eta_+}{k_0}}.
\end{align*}
\]

(22)

Applying the boundary conditions (19)–(21), we may deduce \( A = 0 \), \( E = 0 \) and \( F = 0 \). The coefficient functions \( \{B(\tilde{\eta}), C(\tilde{\eta})\} \) remained undetermined. We believe those undetermined coefficients should be determined by making use of the higher order approximations, and by matching with the inner solution in the near field.

We may continue to perform the asymptotic analysis to the first-order outer solution in a similar manner. We remark here the first-order outer solution is subject to the following governing equations:

\[ k_0^2 \frac{\partial^2 \tilde{\Psi}_1}{\partial \eta_+^2} + 2k_0 k_1 \frac{\partial^2 \tilde{\Psi}_0}{\partial \eta_+^2} + 2k_0 \frac{\partial^2 \tilde{\Psi}_0}{\partial \eta_+ \partial \tilde{\eta}} + k_0' \frac{\partial \tilde{\Psi}_0}{\partial \eta_+} - \frac{k_0 \partial \tilde{\Psi}_0}{\tilde{\eta} \partial \eta_+} = -\tilde{\eta}^2 \tilde{\zeta}_1 - \left( \frac{\partial^2 \tilde{\Psi}_0}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial \tilde{\Psi}_0}{\partial \xi} \right) - \xi^2 \tilde{\zeta}_0, \]

(23)
\[
Pr \left\{ k_0^2 \frac{\partial^2 \tilde{\zeta}_1}{\partial \eta^2} + 2k_0k_1 \frac{\partial^2 \tilde{\zeta}_0}{\partial \eta^2} + 2k_0 \frac{\partial^2 \tilde{\zeta}_0}{\partial \eta \partial \eta} + k_0 \frac{\partial \tilde{\zeta}_0}{\partial \eta} - \frac{k_0}{\eta} \frac{\partial \tilde{\zeta}_0}{\partial \eta} \right\}
\]

\[= - \eta_0^2 k_0 \frac{\partial \tilde{\zeta}_1}{\partial \eta} - \eta_0^2 k_1 \frac{\partial \tilde{\zeta}_0}{\partial \eta} - \frac{\partial^2 \tilde{\zeta}_0}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial \tilde{\zeta}_0}{\partial \xi} \]  

(24)

\[+ \eta_0^2 \left( \frac{\xi}{\partial \bar{\eta}} - \frac{\partial \tilde{\zeta}_0}{\partial \eta} \right) - \frac{\bar{\xi} \bar{\eta}}{T_\infty} \left( \frac{\partial \tilde{T}_1}{\partial \bar{\xi}} + k_0 \frac{\partial \tilde{T}_1}{\partial \eta} + k_1 \frac{\partial \tilde{T}_0}{\partial \eta} + \frac{k_0}{\eta} \frac{\partial \tilde{T}_0}{\partial \eta} \right), \]

\[k_0^2 \frac{\partial^2 \tilde{T}_1}{\partial \eta^2} + 2k_0k_1 \frac{\partial^2 \tilde{T}_0}{\partial \eta^2} + 2k_0 \frac{\partial^2 \tilde{T}_0}{\partial \eta \partial \eta} + k_0 \frac{\partial \tilde{T}_0}{\partial \eta} + \frac{k_0}{\eta} \frac{\partial \tilde{T}_0}{\partial \eta} \]

\[= - \eta_0^2 k_0 \frac{\partial \tilde{T}_1}{\partial \eta} - \eta_0^2 k_1 \frac{\partial \tilde{T}_0}{\partial \eta} \]  

(25)

\[- \left( \frac{\partial^2 \tilde{T}_0}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \tilde{T}_0}{\partial \xi} \right) + \eta_0^2 \left( \frac{\partial \tilde{T}_0}{\partial \xi} - \frac{\partial \tilde{T}_0}{\partial \eta} \right), \]

with the boundary conditions at \( \bar{\eta} \to \infty \) and \( \eta \to \infty \),

\[\bar{\Psi}_1 \to 0, \]

(26)

\[\tilde{\zeta}_1 \to 0, \]

(27)

\[\tilde{T}_1 \to 0. \]

(28)

4. Conclusion

The nature of the phenomenon relating to the dynamical systems in the limit of a small physical parameter has been identified with asymptotic theory. Recently, the physical problem of dendritic growth under the influence of convection motion in the far field from the tip region of the dendrite has been investigated by applying the asymptotic analysis. In this paper, the leading-order asymptotic expansion solutions of the flow field and the temperature field in the limit \( Gr \to 0 \) have been derived. The explicit solutions depend on some undetermined constants as we expected, some higher-order inner and outer asymptotic expansion solutions are apparently needed for matching. Accordingly, the first-order asymptotic governing equations and the corresponding boundary conditions have been formulated. The globally valid asymptotic solution of the dendritic growth under the effect of the buoyancy-driven convection remains unsolved, yet we believe the matched asymptotic solution exists.

References


