Stability Analysis of Mosquito Life Span Model with Delay

D. Pandiaraja
National Centre of Excellence, Statistical and Mathematical Modelling on Bio-Resources Management, Thiagarajar College, Madurai, India.
Department of Mathematics, Thiagarajar College, Madurai, India.

D. Murugeswari and V. Abirami
Department of Mathematics, Thiagarajar College, Madurai, India.

Abstract

In this article, we have incorporated the time delay from mosquitoes searching for oviposition sites into searching for hosts into the nonlinear system of equations [1]. The effect of delay on the stability of the persistent equilibrium and Hopf bifurcation has been investigated. Finally, Numerical simulations have been executed.

Keywords: delay differential equation, stability analysis.

1. Introduction

Mosquitoes are very common fragile insects with an adult life span that lasts about two weeks. A majority of mosquitoes end their life cycle as food for birds, dragonflies and spiders or are killed by the effects of nature such as wind, rain or drought. Blood is the only crucial for the development of mosquito eggs which require certain proteins found in the blood. Mosquitoes are vectors for many of the most important human infections. They can carry malaria, yellow fever, dengue fever and more. Malaria risk is highest in the vicinity of water where mosquitoes oviposit [8].
Mathematical models have been developed to analyze the malaria transmission between humans and mosquitoes with non-linear forces of infection in form of saturated incidence rates [13], Malaria model with stage structured mosquitoes [12], spread of malaria through sensitivity analysis [5], population dynamics with temperature and age dependent survival [3], breaking the life cycle of a mosquito that incorporate a time delay at the larval stage that accounts for the period of growth and development to pupa [2], estimation of seasonal variation of mosquito population [11], mosquito dispersal in a heterogeneous environment [1].

Stability of a mathematical model of malaria transmission [10], a delayed Ross-Macdonald model for malaria transmission [6], mosquito dispersal in a heterogeneous environment [1], predator-prey populations subjected to constant effort of harvesting [9], a ratio dependent predator-prey model with quadratic harvesting [14], HIV model [7], HIV primary infection [15] have been discussed. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission [16] has been detailed. Stability and bifurcation of a two-species Leslie-Gower predator-prey system with time delay [4] has been detailed.

Let us consider the following mathematical model to investigate the impact of dispersal and heterogeneous distribution of resources on the distribution and dynamics of mosquito populations [1].

\[
\begin{align*}
\frac{dE}{dT} &= b \rho_{L_1} A_o - (\mu_E + \rho_E)E \\
\frac{dL}{dT} &= \rho_E E - (\mu_{L_1} + \mu_{L_2} L + \rho_L)L \\
\frac{dP}{dT} &= \rho_L L - (\mu_P + \rho_P)P \\
\frac{dA_h}{dt} &= \rho_P P + \rho_{A_h} A_o - (\mu_{A_h} + \rho_{A_h})A_h \\
\frac{dA_r}{dt} &= \rho_{A_h} A_h - (\mu_{A_r} + \rho_{A_r})A_r \\
\frac{dA_o}{dt} &= \rho_{A_r} A_r - (\mu_{A_o} + \rho_{A_o})A_o
\end{align*}
\]

where, 
- \( b \) - number of female eggs laid per oviposition.
- \( \rho_E \) - egg hatching rate into larvae (\( day^{-1} \)).
- \( \rho_L \) - rate at which larvae developing into pupae (\( day^{-1} \)).
- \( \rho_P \) - rate at which pupae develope into adult or emergence rate (\( day^{-1} \)).
- \( \mu_E \) - egg mortality rate (\( day^{-1} \)).
- \( \mu_P \) - pupae mortality rate (\( day^{-1} \)).
- \( \mu_{L_1} \) - density independent larvae mortality rate (\( day^{-1} \)).
- \( \mu_{L_2} \) - density dependent larvae mortality rate (\( day^{-1} \)).
- \( \rho_{A_h} \) - rate at which host-seeking mosquitoes enter the resting state (\( day^{-1} \)).
- \( \rho_{A_r} \) - rate at which resting mosquitoes enter oviposition site searching state (\( day^{-1} \)).
\( \rho_{Ao} \) - oviposition rate \((day^{-1})\).
\( \mu_{Ah} \) - mortality rate of mosquitoes of searching for hosts \((day^{-1})\).
\( \mu_{Ar} \) - mortality rate of resting mosquitoes \((day^{-1})\).
\( \mu_{Ao} \) - mortality rate of mosquito searching for oviposition sites \((day^{-1})\).

The population reproduction number \( R_0 \) is
\[
R_0 = \frac{b \prod_j (\frac{\rho_j}{\mu_j + \rho_j})}{1 - \prod_i (\frac{\rho_{Aj}}{\mu_{Ai} + \rho_{Ai}})}
\]
where \( j = E, L, P, A_h, A_r, A_o \) and \( i = h, r \) and \( o \).

We have incorporated the time delay \((\tau)\) from mosquitoes searching for oviposition sites into searching for hosts into the above model. Thus, we have the following delay model:

\[
E' = b \rho_{Ar} A_o - (\mu_E + \rho_E) E \\
L' = \rho_E E - (\mu_{L1} + \mu_{L2} L + \rho_L) L \\
P' = \rho_L L - (\mu_P + \rho_P) P \\
A_h' = \rho_P P + \rho_{Ah} A_o(t - \tau) - (\mu_{Ah} + \rho_{Ah}) A_h \\
A_r' = \rho_{Ah} A_h - (\mu_{Ar} + \rho_{Ar}) A_r \\
A_o' = \rho_{Ar} A_r - \mu_{Ao} A_o - \rho_{Ao} A_o(t - \tau)
\]

In section 2, stability analysis of the system of equations (1.7)–(1.12) have been discussed at persistent equilibrium
\( P_e = (E^*, L^*, P^*, A_h^*, A_r^*, A_o^*) \),
where
\[
E^* = \frac{b \rho_{Ah} A_o^*}{\mu_E + \rho_E}, \\
L^* = \frac{\mu_E L^*}{\mu_{L1} + \rho_L (R_0 - 1)}, \\
P^* = \frac{\rho_L L^*}{\mu_P + \rho_P}, \\
A_h^* = \frac{\rho_P P^* R_0}{(\mu_{Ah} + \rho_{Ah}) B_1}, \\
A_r^* = \frac{\rho_{Ah} A_h^*}{\mu_{Ar} + \rho_{Ar}}, \\
A_o^* = \frac{\rho_{Ar} A_r^*}{\mu_{Ao} + \rho_{Ao}}
\]
where
\[
R_0 = \frac{B_1}{1 - \prod_i (\frac{\rho_{Ai}}{\mu_{Ai} + \rho_{Ai}})}.
\]
\[ B_1 = b \prod_{j} \left( \frac{\rho_j}{\mu_j + \rho_j} \right), \]

\( j = E, L, P, A_h, A_r, A_o \) and \( i = A_h, A_r, A_o. \)

In section 3, numerical simulations have been executed.

2. Stability analysis

**Theorem 2.1.** If \( R_0 < 1 \), then the mosquito free equilibrium point of system (1.7)-(1.12) is locally asymptotically stable for all \( \tau \geq 0 \).

Now we investigate the effect of the time delay on the stability of the persistent equilibrium \( P_e \). The required jacobian matrix at \( P_e \) is given by,

\[
J_{P_e} = \begin{pmatrix}
-(\mu_E + \rho_E) & 0 & 0 & 0 & 0 & b\rho_{A_o} \\
\rho_E & -(\mu_{L_1} + \rho_L) - \phi & 0 & 0 & 0 & 0 \\
0 & \rho_L & -(\mu_P + \rho_P) & 0 & 0 & 0 \\
0 & 0 & \rho_P & -(\mu_{A_h} + \rho_{A_h}) & 0 & \rho_{A_o}e^{-\lambda \tau} \\
0 & 0 & 0 & \rho_{A_h} & -(\mu_{A_r} + \rho_{A_r}) & 0 \\
0 & 0 & 0 & 0 & \rho_{A_r} & -(\mu_{A_o} + \rho_{A_o}e^{-\lambda \tau})
\end{pmatrix}
\]

where, \( \phi = 2(\mu_{L_1} + \rho_L)(R_0 - 1) \).

To evaluate the eigenvalues of \( J_{P_e} \), we solve \( det(J_{P_e} - \lambda I) = 0 \). We use the concept of block matrices to obtain this determinant.

Let \( J = J_{P_e} - \lambda I \) be a block matrix given by,

\[
J = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

with the following components:

\[
A = \begin{pmatrix}
-(\mu_E + \rho_E) - \lambda & 0 & 0 \\
\rho_E & -(\mu_{L_1} + \rho_L) - \phi - \lambda & 0 \\
0 & \rho_L & -(\mu_P + \rho_P) - \lambda
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & 0 & b\rho_{A_o} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
0 & 0 & \rho_P \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and,

\[
D = \begin{pmatrix}
-(\mu_{A_h} + \rho_{A_h}) - \lambda & 0 & \rho_{A_o}e^{-\lambda \tau} \\
\rho_{A_h} & -(\mu_{A_r} + \rho_{A_r}) - \lambda & 0 \\
0 & \rho_{A_r} & -(\mu_{A_o} + \rho_{A_o}e^{-\lambda \tau}) - \lambda
\end{pmatrix}
\]
Therefore the concepts of block matrices that $det(J) = det(AD - BC)$. Since BC is a zero matrix, then $det(J) = det(AD) = 0$. Then, the characteristic equation is,

$$
\lambda^6 + \lambda^5(A + a + b) + \lambda^4(B + A(a + b) + ab + c) \\
+ \lambda^3(C + (a + b)B + A(ab + c) + ac) \\
+ \lambda^2(C(a + b) + B(ab + c) + Aac + \lambda(C(ab + c) + Bac) + acC) \\
+ \left[\lambda^5D + \lambda^4(E + (a + b)D) + \lambda^3(F + (a + b)E + D(ab + c))
\right. \\
\left. + \lambda^2((a + b)F + E(ab + c) + acD) + \lambda(F(ab + c) + Eac) + acF\right]e^{-\lambda\tau} = 0
$$

where,

$$
\begin{align*}
a &= \mu_E + \rho_E, \quad b = \mu_L + \rho_L + \phi + \mu_P + \rho_P, \\
c &= (\mu_L + \rho_L + \phi)(\mu_P + \rho_P), \\
A &= \mu_{Ah} + \rho_{Ah} + \mu_{Ar} + \rho_{Ar}, \\
B &= (\mu_{Ah} + \rho_{Ah})(\mu_{Ah} + \rho_{Ah} + \mu_{Ar} + \rho_{Ar}), \\
C &= \mu_{Ah} + \rho_{Ah}(\mu_{Ah} + \rho_{Ah}), \\
D &= \rho_{Ah}, \quad E = \rho_{Ah} + \mu_{Ah} + \mu_{Ar} + \rho_{Ar}, \\
F &= \rho_{Ah}(\mu_{Ah} + \rho_{Ah})(\mu_{Ar} + \rho_{Ar}) - \rho_{Ah}\rho_{Ah}\rho_{Ar}.
\end{align*}
$$

Then,

$$
P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \quad (2.1)
$$

where

$$
P(\lambda) = \lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6
$$

$$
Q(\lambda) = b_1\lambda^5 + b_2\lambda^4 + b_3\lambda^3 + b_4\lambda^2 + b_5\lambda + b_6
$$

where

$$
\begin{align*}
a_1 &= A + (a + b), \quad a_2 = B + A(a + b) + ab + c, \\
a_3 &= C + (a + b)B + A(ab + c) + ac, \quad a_4 = C(a + b) + B(ab + c) + Aac, \\
a_5 &= C(ab + c) + Bac, \quad a_6 = acC, \quad b_1 = D, \\
b_2 &= E + (a + b)D, \quad b_3 = F + (a + b)E + D(ab + c), \\
b_4 &= (a + b)F + E(ab + c) + acD, \quad b_5 = F(ab + c) + Eac, \quad b_6 = acF.
\end{align*}
$$

If $\tau = 0$, then the above characteristic equation becomes,

$$
\lambda^6 + (a_1 + b_1)\lambda^5 + (a_2 + b_2)\lambda^4 + (a_3 + b_3)\lambda^3 + (a_4 + b_4)\lambda^2 + (a_5 + b_5)\lambda + (a_6 + b_6) = 0
$$
Theorem 2.2. If \( R_0 > 1 \) and \( p > 0, \ pq > r, \ r(pq - r) > p(ps - t) \), and \((ps - t) [(pq - r)r - p(ps - t)] > (pq - r) [((pq - r)t - up^2) \), then the persistent equilibrium \( P_e \) is locally asymptotically stable for all delay \( \tau \geq 0 \).

Proof. If \( R_0 > 1 \) and \( \tau > 0 \), assuming \( \lambda = i \omega \) with \( \omega > 0 \) in (2.1) we get,

\[
- \omega^6 + ia_1 \omega^5 + a_2 \omega^4 - ia_3 \omega^3 - a_4 \omega^2 + ia_5 \omega + a_6
\]

\[
= [ib_1 \omega^5 + b_2 \omega^4 - ib_3 \omega^3 - b_4 \omega^2 + ib_5 \omega + b_6](\cos \omega \tau - i \sin \omega \tau) = 0
\]

Separating the real and imaginary parts, we get,

\[
- \omega^6 + a_2 \omega^4 - a_4 \omega^2 + a_6 = (-b_2 \omega^4 + b_4 \omega^2 - b_6) \cos \omega \tau
\]

\[
+ (-b_1 \omega^5 + b_3 \omega^3 - b_5 \omega) \sin \omega \tau
\]

(2.2)

\[
a_1 \omega^5 - a_3 \omega^3 + a_5 \omega = (-b_1 \omega^5 + b_3 \omega^3 - b_5 \omega) \cos \omega \tau
\]

\[
- (-b_2 \omega^4 + b_4 \omega^2 - b_6) \sin \omega \tau
\]

(2.3)

Squaring and adding (2.2) and (2.3),

\[
\omega^{12} + \omega^{10}(a_1^2 - 2a_2 - b_1^2) + \omega^8(a_2^2 + 2(a_4 - a_1 a_3 + b_1 b_3) - b_2^2) + \omega^6(a_3^2 + 2(a_1 a_5 - a_6 - a_2 a_4 - b_1 b_5 + b_2 b_4) - b_3^2) + \omega^4(a_4^2 + 2(a_2 a_6 - a_3 a_5 - b_2 b_6 + b_3 b_5) - b_4^2) + \omega^2(a_5^2 + 2(b_4 b_5 - a_4 a_6) - b_5^2) + (a_6^2 - b_6^2) = 0
\]

(2.4)

Let

\[
z = \omega^2, \quad p = a_1^2 - 2a_2 - b_1^2,
\]

\[
q = a_2^2 + 2(a_4 - a_1 a_3 + b_1 b_3) - b_2^2,
\]

\[
r = a_3^2 + 2(a_1 a_5 - a_6 - a_2 a_4 - b_1 b_5 + b_2 b_4) - b_3^2,
\]

\[
s = a_4^2 + 2(a_2 a_6 - a_3 a_5 - b_2 b_6 + b_3 b_5) - b_4^2,
\]

\[
t = a_5^2 + 2(b_4 b_5 - a_4 a_6) - b_5^2
\]

and \( u = a_6^2 - b_6^2 \). Thus, we have

\[
G(z) = z^6 + pz^5 + qz^4 + rz^3 + sz^2 + tz + u = 0
\]

(2.5)

Since \( p > 0, \ pq > r, \ r(pq - r) > p(ps - t) \) and \((ps - t) [(pq - r)r - p(ps - t)] > (pq - r) [((pq - r)t - up^2) \), then \( z = \omega^2 < 0 \). So our assumption that \( \lambda = i \omega \) is a root of (2.1) is wrong which means that (2.1) has no positive roots and the real parts of all eigenvalues are negative for all delay \( \tau \geq 0 \).
3. Hopf Bifurcation

If \( u < 0 \), then \( G(0) = u < 0 \) and \( \lim_{\tau \to \infty} G(\tau) = \infty \). Then there exists at least one positive root satisfying equation (2.1), so the characteristic equation (2.1) has a pair of purely imaginary roots of the form \( \pm i \omega_0 \). Eliminating \( \sin \omega \tau \) from (2.2) and (2.3), we have

\[
\cos \omega \tau = \frac{[(-b_2 \omega^4 + b_4 \omega^2 - b_6)(-\omega^6 + a_2 \omega^4 - a_4 \omega^2 + a_6) + (-b_1 \omega^5 + b_3 \omega^3 - b_5 \omega)(a_1 \omega^5 - a_3 \omega^3 + a_5 \omega)]}{(-b_2 \omega^4 + b_4 \omega^2 - b_6)^2 + (-b_1 \omega^5 + b_3 \omega^3 - b_5 \omega)^2}
\]

Therefore, \( \tau^*_n \) corresponding to \( \omega_0 \) is given by;

\[
\tau^*_n = \frac{1}{\omega_0 \cos^{-1} \left( \frac{[(-b_2 \omega^4 + b_4 \omega^2 - b_6)(-\omega^6 + a_2 \omega^4 - a_4 \omega^2 + a_6) + (-b_1 \omega^5 + b_3 \omega^3 - b_5 \omega)(a_1 \omega^5 - a_3 \omega^3 + a_5 \omega)]}{(-b_2 \omega^4 + b_4 \omega^2 - b_6)^2 + (-b_1 \omega^5 + b_3 \omega^3 - b_5 \omega)^2} + \frac{2n\pi}{\omega_0} \right)}
\]

If \( \tau = 0 \), the persistent equilibrium \( P_e \) is stable when \( R_0 > 1 \) [1]. Therefore \( P_e \) is stable for \( \tau < \tau_0 \) where \( \tau_0 = \tau^*_n \) as \( n = 0 \). Hence, if (2.1) has a pair of purely imaginary roots, then it has roots with positive real part (by continuity in \( \tau \)). Thus, \( P_e \) becomes unstable and periodic solutions may happen that is Hopf bifurcation occur if \( \frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=\tau_0} > 0 \).

Differentiating (2.1) with respect to \( \tau \), we get

\[
[(6\lambda^5 + 5a_1 \lambda^4 + 4a_2 \lambda^3 + 3a_3 \lambda^2 + 2a_4 \lambda + a_5) + e^{-\lambda \tau} (5b_1 \lambda^4 + 4b_2 \lambda^3 + 3b_3 \lambda^2 + 2b_4 \lambda + b_5) - \tau e^{-\lambda \tau} (b_1 \lambda^5 + b_2 \lambda^4 + b_3 \lambda^3 + b_4 \lambda^2 + b_5 \lambda + b_6)] \frac{d\lambda}{d\tau} = \lambda e^{-\lambda \tau} (b_1 \lambda^5 + b_2 \lambda^4 + b_3 \lambda^3 + b_4 \lambda^2 + b_5 \lambda + b_6)
\]

Then

\[
\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{5\lambda^6 + 4a_1 \lambda^5 + 3a_2 \lambda^4 + 2a_3 \lambda^3 + a_4 \lambda^2 - a_6}{-\lambda^2(\lambda^6 + a_1 \lambda^5 + a_2 \lambda^4 + a_3 \lambda^3 + a_4 \lambda^2 + a_5 \lambda + a_6)} + \frac{4b_1 \lambda^5 + 3b_2 \lambda^4 + 2b_3 \lambda^3 + b_4 \lambda^2 - b_6}{\lambda^2(b_1 \lambda^5 + b_2 \lambda^4 + b_3 \lambda^3 + b_4 \lambda^2 + b_5 \lambda + b_6)} \frac{\tau}{\lambda}
\]
Thus

\[
\text{sign} \left\{ \frac{d(Re\lambda)}{d\tau} \right\}_{\lambda=i\omega_0} = \text{sign} \left\{ \frac{\text{Re} \left( \frac{d\lambda}{d\tau} \right)}{\lambda} \right\}_{\lambda=i\omega_0}
\]

\[
= \text{sign} \left\{ \frac{1}{\omega_0^2} \left[ \frac{5\omega_0^{12} + 4\omega_0^{10}(a_1^2 - 2a_2 - b_1^2) + 3\omega_0^8(a_2^2 + 2(a_4 - a_1a_3 + b_1b_3) - b_2^2)}{(b_2\omega_0^4 - b_4\omega_0^2 + b_6)^2 + (b_1\omega_0^5 - b_3\omega_0^3 + b_5\omega_0)^2}
\right. \\
+ \frac{2\omega_0^6(a_3^2 + 2(a_1a_5 - a_6 - a_2a_4 - b_1b_5 + b_2b_4) - b_3^2)}{(b_2\omega_0^4 - b_4\omega_0^2 + b_6)^2 + (b_1\omega_0^5 - b_3\omega_0^3 + b_5\omega_0)^2}
\left. \\
+ \frac{\omega_0^4(a_4^2 + 2(a_2a_6 - a_3a_5 - b_2b_6 + b_3b_5) - b_5^2) + (b_6^2 - a_6^2)}{(b_2\omega_0^4 - b_4\omega_0^2 + b_6)^2 + (b_1\omega_0^5 - b_3\omega_0^3 + b_5\omega_0)^2} \right] \right)
\]

If

\[
(a_1^2 - 2a_2 - b_1^2) > 0,
\]
\[
a_2^2 + 2(a_4 - a_1a_3 + b_1b_3) - b_2^2 > 0,
\]
\[
a_3^2 + 2(a_1a_5 - a_6 - a_2a_4 - b_1b_5 + b_2b_4) - b_3^2 > 0,
\]
\[
a_4^2 + 2(a_2a_6 - a_3a_5 - b_2b_6 + b_3b_5) - b_4^2 > 0,
\]
\[
b_6^2 - a_6^2 > 0
\]

which means that \( p > 0, q > 0, r > 0, s > 0 \) and \( u < 0 \), then

\[
\left. \frac{d(Re\lambda)}{d\tau} \right|_{\tau=\tau_0, \omega=\omega_0} > 0.
\]

Hence we have at least one eigenvalue with positive real part for \( \tau > \tau_0 \) and the conditions for Hopf bifurcation are satisfied yielding periodic solutions at \( \tau = \tau_0, \omega = \omega_0 \). Thus we have the following theorem:

**Theorem 3.1.** If \( R_0 > 1 \) and \( p > 0, q > 0, r > 0, s > 0 \) and \( u < 0 \), the persistent equilibrium \( P_e \) remains stable for \( \tau < \tau_0 \) and unstable when \( \tau > \tau_0 \), a Hopf bifurcation occurs as \( \tau \) passes through \( \tau_0 \).

### 4. Numerical Simulation

In this section we present numerical results of the system (1.7)-(1.12) to verify the analytical predictions obtained in the previous section. Let us consider the system with the parameter values \( b = 100, \rho_E = 0.50, \rho_L = 0.14, \rho_p = 0.50, \mu_E = 0.39, \mu_L_1 = 0.44, \mu_L_2 = 0.05, \mu_p = 0.37, \rho_{A_b} = 0.46, \rho_{A_c} = 0.43, \rho_{A_0} = 3.0, \mu_{A_b} = 0.18, \mu_{A_c} = 0.0043 \) and \( \mu_{A_0} = 0.41 \). So the system (1.7)-(1.12) has a positive equilibrium \( P_e(1480.19, 116, 18.67, 39.15, 41.47, 5.23) \).
5. Conclusion

We investigated the effect of time delay and Hopf bifurcation in the mosquito life cycle model. It is observed that if $R_o < 1$ the mosquito free equilibrium point of system is locally asymptotically stable for all $\tau \geq 0$ and if $R_o > 1$ the persistent equilibrium is locally asymptotically stable for all $\tau \geq 0$.

Acknowledgement

We are grateful to Ministry of Human Resource Development, Government of India, New Delhi and The Management, Thiagarajar College, Madurai, Tamilnadu for rendering financial and other facilities.

References


