Mathematical Modeling of Unsteady Flow Through A Tube With Time Dependent Stenosis in Hemodynamics

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Abstract

In this paper, a mathematical model is developed, taking into consideration the slip velocity at the wall of blood vessel of time-dependent Stenosis with unsteady flow through a tube in the presence of a Stenosis. Momentum integral method has been used to evaluate axial velocity and pressure gradient in time dependent Stenosis. Results have also been compared with and without the effect of slip velocity. It has been observed that axial velocity differ significantly with existence of slip velocity. The present study asserts that the slip velocity has a reducing effect on the pressure drop.

Keywords : Unsteady Flow, Integral Momentum Method, slip velocity, stenosis.

Introduction

Fat and cholesterol deposits on the inner walls of arteries cause narrowing of the lumen diameter, resulting in a decrease in blood flow to the region upstream of the constriction. Depending on the artery affected, the reduction in flow may result in stroke, gangrene, angina, or even a heart attack. Certain hydro dynamical factors play a very important role in the development of this disease. Lee and Fung [1] employed numerical technique to solve a problem concerning blood flow through a stenosed tube. Haldar [2] had obtained analytical results for oscillatory flow of blood which behaves as a Newtonian fluid.

McDonald [3] remarked that for vessels of radius greater than 0.25 mm., blood may be considered as a homogenous Newtonian fluid. At low shear rates blood...

In the present paper, a mathematical model has been developed and it has been shown that pressure drop in constricted tube increases as time increases but it reduces as slip-velocity increases.

**Stenosis Model**

By Mishra and Kar [8], it is obvious that Stenosis (constriction) has no well defined geometrical configuration. In general, complex three-dimensional flow patterns have been developed near the stenosis which is virtually impossible to analyse. In this paper, 'collar like' stenosis model i.e. axisymmetric constriction in a long tube has been considered. It is true that the size of stenosis increases with time and attains some fixed geometrical configuration after some time. It is assumed that flow is unsteady and laminar, the artery is of constant diameter \(2R_o\) preceding and following the stenosis.

Fig. 1 describes the stenosis geometry in the cylindrical polar coordinate system. It is further assumed that stenosis grows in an axially symmetric manner due to abnormal growth over a length \(2L_o\) of the artery. Tondon and Katiyar [11] expressed the local radius \(R\) of the axisymmetric tube as a function of longitudinal coordinate \(x\) and time \(t\) as

\[
R = R_o - \frac{\delta}{2} \left(1 - e^{-\frac{t}{T}}\right) \left(1 + \cos \frac{\pi x}{L_o}\right).
\]  

(1)
where \( R = R_o \) for \( t = 0 \), \( \delta \) is constant (height of stenosis). \( \frac{t}{T} \) is time function and \( T \) is time constant for stenotic growth. However, it should be noted that although the growth rate as characterized by the parameter, \( T \), is not important to the fluid mechanics of the problem, the rates at which variables such as pressure and shearing stress are changing may play an important role in certain cellular process.

**Analysis**

For two-dimensional flow, the governing equations for the flow of a Newtonian, viscous, incompressible fluid may be taken in the form in axial direction

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial x^2} \right) \tag{2}
\]

and in the radial direction

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial x^2} \right) \tag{3}
\]

while the continuity equation is

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{4}
\]

where \( u \) represents the axial velocity, \( v \) the radial velocity, \( \rho \) the density, \( p \) the pressure and \( \nu \) the kinematic viscosity coefficient of the blood.

During the initial development of the stenosis \( \frac{\delta}{R_0} \ll 1 \) and \( v \ll u \) then radial variation of pressure i.e. \( \frac{\partial p}{\partial r} \) may be neglected and \( \frac{\delta}{L_0} \ll 1 \) then normal stress gradient \( \frac{\partial^2 u}{\partial x^2} \) is negligible compared to the stress gradient \( \frac{\partial^2 u}{\partial r^2} \).

The governing equations can be approximated for mild stenosis as

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \tag{5}
\]

and \( \frac{\partial p}{\partial r} = 0 \). \tag{6}

By applying the integral momentum method and using the continuity equation, integrating equation (5) over the cross-section of vessel, we obtain the integral equation as
\[ \frac{\partial}{\partial t} \int_0^R ru \, dr + \frac{\partial}{\partial x} \int_0^R ru^2 \, dr = -\frac{1}{\rho} \left( \frac{R^2}{2} \right) \frac{\partial p}{\partial x} + \nu R \left( \frac{\partial u}{\partial r} \right)_R. \]  

(7)

At the surface, the boundary condition is \( v = 0 \) and \( u = W \) (slip velocity) at \( r = R \).

The volume flux is given by

\[ Q = 2\pi \int_0^R ru \, dr, \]  

(8)

we choose the velocity profile in the form of as

\[ \hat{u} = \frac{u}{U} = A + B \eta + C \eta^2 + D \eta^3 + E \eta^4, \]  

(9)

where \( \eta = \left( \frac{R - r}{R} \right) \) \( (9a) \)

\( U \) being the centre line velocity and \( A, B, C, D \) and \( E \) are constants to be determined from the velocity constraints. Using (9) and (9a), equation (8) may be rewritten as

\[ Q = 2\pi R^2 U \int_0^1 (1 - \eta) \hat{u} \, d\eta. \]  

(9b)

The appropriate boundary conditions are

\[ u = U \quad \text{at} \quad r = 0, \]  

(10a)

\[ u = W \quad \text{at} \quad r = R, \]  

(10b)

\[ \hat{u} = \frac{u}{U} \]  

(10c)

\[ \frac{\partial^2 \hat{u}}{\partial r^2} = -\frac{2}{R^2} \frac{U}{R} \]  

(10d)

together with

\[ \frac{\partial \hat{p}}{\partial x} = \frac{\mu}{r \frac{\partial}{\partial r}} \left( \frac{\partial \hat{u}}{\partial r} \right) \]  

(10e)

The first one describes the centre line velocity, second is the slip condition at the wall, third describes the viscous stress which is proportional to \( \frac{\partial u}{\partial r} \) and which should approach zero as the radius of the tube approaches zero when other forces are finite.

The second radial derivative of \( u \) at \( r = 0 \) may be approximated by the fourth condition assuming the velocity profile to be nearly parabolic at the axis as represented by the Poiseuille's profile \( \hat{u} = 1 - \left( r / R \right)^2 \). Finally, the fifth condition represents the validity of equation (5) at \( r = R \). The above written condition transformed in terms of variable \( \eta \) introduced in (9a) may be put as

\[ \hat{u} = 1 \quad \text{at} \quad \eta = 1, \]  

(11a)

\[ \hat{u} = \frac{W}{U} \quad \text{at} \quad \eta = 1, \]  

(11b)

\[ \frac{\partial \hat{u}}{\partial \eta} = 0 \quad \text{at} \quad \eta = 1, \]  

(11c)
\[
\frac{\partial^2 \hat{u}}{\partial \eta^2} = -2 \quad \text{at} \quad \eta = 1, \quad \text{(11d)}
\]

and

\[
\frac{\partial p}{\partial x} = \frac{\mu U}{R^2} (1-\eta) \frac{\partial}{\partial \eta} \left[ (1-\eta) \frac{\partial \hat{u}}{\partial \eta} \right] \quad \text{at} \quad \eta = 0, \quad \text{(11e)}
\]

Using conditions (11a) to (11e), the velocity profile is as follows

\[
\hat{u} = \frac{U}{R} \left( \frac{-\lambda + 10 - 12g}{7} \right) \eta + \left( \frac{3\lambda + 5 - 6g}{7} \right) \eta^2 + \left( \frac{-3\lambda - 12 + 20g}{7} \right) \eta^3
\]

\[
+ \left( \frac{\lambda + 4 - 9g}{7} \right) \eta^4 + g,
\]

where

\[
\lambda = \frac{R^2}{2} \frac{d}{p} \frac{d}{x} \quad \text{and} \quad g = \frac{W}{U}.
\]

Here the velocity profile becomes a function of only one parameter \( \lambda \), which is function of pressure gradient \( \frac{\partial p}{\partial x} \).

Substituting (12) into (9b) and taking the integration we get

\[
U = \frac{210}{97\pi R^2} \left[ Q + \frac{\pi R^2}{2} \frac{d}{p} \frac{d}{x} + \frac{78\pi R^2 W}{210} \right]
\]

The parameter \( \lambda \) may be determined from equation (7) as

\[
\lambda = \frac{4}{5} \left( 6g - 5 \right) - \frac{14}{56U} \frac{\partial}{\partial t} \left[ R^2 \int_0^1 (1-\eta) u d\eta \right] - \frac{14}{56U} \frac{\partial}{\partial x} \left[ R^2 \int_0^1 (1-\eta) u^2 d\eta \right], \quad \text{(15)}
\]

In equation (15), neglecting the terms in velocity profile, higher than two because profile according to Bugliarello [9] due to specific diameter (5.0 \( \mu m \)) of the arteries is as

\[
u = 2\overline{U} \left[ 1 - (r/R)^2 \right],
\]

where

\[
\overline{U} = \frac{(R^2/8\mu) d}{d x} p,
\]

\( \overline{U} \) is the average velocity of cross-section and \( \frac{d p}{d x} < 0 \).

Substitution of (16) into momentum integral equation (7) yields

\[
\frac{\partial}{\partial t} \left( \frac{\overline{U} R^2}{2} \right) + \frac{\partial}{\partial x} \left( \frac{7}{3} \frac{U^2}{R^2} R^2 \right) = \frac{1}{\rho} \left( \frac{R^2}{2} \right) \frac{d}{d x} p + \frac{\sqrt{U}}{7} \left( \lambda - 10 + 12g \right)
\]

Using \( \overline{U} = Q/\pi R^2 \) and combining the resulting equation with equation (14).

The pressure gradient is obtained in the form

\[
\frac{d}{d x} = 1.5 \frac{\mu W}{R^2} - 8 \frac{\mu U}{R^2} - 2.9 \frac{\overline{U} R^2}{R} \frac{\partial R}{\partial t} - 3.7 \frac{\overline{U} R^2}{R} \frac{\partial R}{\partial x}.
\]

The first term on the right hand side of equation (19) is due to the slip velocity, the second term is due to the viscous shearing stress, third due to rate of
change with time and fourth term is due to inertia of blood.

In non-dimensional form, equation (19) read as

\[
\left( \frac{R_0}{U_0 \rho} \right) \frac{d \mu}{dx} = \frac{3}{Re_0} \left( \frac{R_0}{R} \right)^2 \frac{W}{U_0} - 16 \left( \frac{R_0}{R} \right)^2 - 2.9 \left( \frac{R_0}{R} \right) \frac{1}{U_0} \frac{\partial R}{\partial t} - 3.7 \left( \frac{R_0}{R} \right) \frac{\partial R}{\partial x}.
\]

(20)

Where \( Re_0 = \left( \frac{2R_0 U_0 \rho}{\mu} \right) \)

(21)

is the Reynolds number upstream from the stenosis, \( U_0 \) being the average velocity.

Substituting (14) and (19) into (12), we obtain the velocity distribution \( u \) as a function of \( r \) and \( x \) as follows

\[
\hat{u} = \frac{u}{U_0} = \left( 1.4 \eta - 5.6 \eta^2 + 5.9 \eta^3 - 2 \eta^4 \right) \frac{Re_0}{7} \left( \frac{R}{R_0} \right) \frac{\partial R}{\partial x}
\]

\[
+ \left( 1.2 \eta - 4.7 \eta^2 + 4.9 \eta^3 - 1.6 \eta^4 \right) \frac{Re_0}{7} \left( \frac{R}{R_0} \right) \frac{\partial R}{\partial t}
\]

\[
+ \frac{W}{7U_0} \left( 7 - 5.5 \eta + 2.5 \eta^2 + 5.5 \eta^3 - 4.5 \eta^4 \right)
\]

\[
+ \left( 22 \eta + 11 \eta^2 - 26.4 \eta^3 + 8.8 \eta^4 \right) \frac{1}{7} \left( \frac{R_0}{R} \right)^2
\]

\[
+ \frac{1}{7} \left( 6 \eta - 2.5 \eta^2 + 26.4 \eta^3 - 8.8 \eta^4 \right)
\]

(22)

The skin-friction \( \tau_w \) is given by

\[
\tau_w = \mu \left( \frac{\partial u}{\partial r} \right)_{r=R}
\]

(23)

Employing (22), we further derive the following expression for the skin-friction

\[
\frac{\tau_w}{\rho U_0^2} = \frac{60}{7} \left( \frac{R_0}{R} \right) \frac{1}{Re_0} \frac{W}{U_0} - 0.4 \frac{\partial R}{\partial x} - 2.4 \frac{\partial R}{\partial t} - 44 \frac{1}{7} \left( \frac{R_0}{R} \right)^3 - 12 \frac{1}{7} \left( \frac{R_0}{R} \right)(24)
\]

Results and Discussion

The slip velocity has been taken to be equal to 10% the average velocity of the blood in a normal artery. The calculation has been carried out at the locations defined by

\[
\frac{x}{L_0} = 0, \pm 0.5 \quad \text{and} \quad \pm 1.0 \quad \text{for three different values of Reynolds number, viz.} \ Re_0 = 100, 300 \quad \text{and} \quad 500 \quad \text{for different time parameters} \ \frac{t}{T} = 0 \quad \text{and} \quad 3.
\]

Figs. 2 and 3 depict the variation of the non-dimensional axial velocity of blood flow in the stenosed arterial segment with and without consideration of slip velocity. The variation of the non-dimensional axial velocity of blood flow in the stenosed arterial segment with and without consideration of the slip velocity has been depicted through figs. 2, 3, 4, 5 and 6, 7 for \( Re_0 = 100, 300 \quad \text{and} \quad 500 \) respectively for different values of time parameter. It may be observed that with the increase in
Reynolds number, the axial velocity in the presence of slip velocity at the well
appreciably differ from the corresponding velocity in the absence of slip velocity. The
difference between the velocity with slip velocity and velocity without slip is
decreasing. Hence axial velocities are also increasing with time and Reynolds
numbers at $\frac{x}{l_0} = 0, \pm 0.5$ and $\pm 1.0$.

**Conclusion**

We shall use the following geometrical configuration of the stenosis, used by Young

$$ R = R_0 - \frac{\delta}{2} \left( 1 - e^{-\frac{x}{\pi l_0}} \right) \left( 1 + \cos \frac{\pi x}{l_0} \right). $$

(26)

The stenosis parameter $R_0$, $l_0$, and $\delta$ are taken to be related as

$$ l_0 = 4R_0 = 12\delta. $$

(27)

The pressure drop in its non-dimensional form may be obtained by integrating equation (20) from the upstream end of stenosis to any point along it, is given by

$$ \frac{p_0 - p}{\rho U_0^2} = \frac{1}{R_0} \int_{-l_0}^{l_0} \frac{R_0}{R} \left( \frac{\delta}{R} \right)^2 \frac{32U_0}{R_0 \rho U_0} \int_{-l_0}^{l_0} \left( \frac{R_0}{R} \right) \frac{\partial \left( \frac{R_0}{R} \right)}{\partial x} dx + \frac{9}{16} \left( \frac{R_0}{R} \right) \int_{-l_0}^{l_0} \left( \frac{R_0}{R} \right) dx - \frac{9}{4} \frac{W}{R_0 \rho U_0} \left( \frac{R_0}{R} \right) \int_{-l_0}^{l_0} \left( \frac{R_0}{R} \right) dx. $$

(28)

where $p_0$ represents the pressure at the upstream end $x = -l_0$ of the stenosis. Table 1 gives the values of the non-dimensional pressure drop for $Re_0 = 2U_0 R_0 / \nu = 500$ at different locations of the stenosed artery for four different values of the parameter for different values of $\frac{l}{T} = 0, 1, 2$, and 3. T The values obtained by Forrester and Young [8] have also been shown for comparison on idea of the extent to which the slip velocity affects the pressure drop.

Dimensionless pressure drops between two axial positions one upstream and
another downstream of stenosis are obtained by integrating equation (20) from

$$ x = -4l_0 \text{ to } x = 4l_0 $$

$$ \frac{\Delta p}{\rho U_0^2} = \int_{-4l_0}^{4l_0} \left( \frac{R_0}{R} \right)^2 dx, $$

(29)

where

$$ \frac{R_0}{R} = \frac{6}{5 + e^{-\frac{x}{l_0}} - (1 - e^{-\frac{x}{l_0}}) \cos \frac{\pi x}{4l_0}}. $$

For a straight tube, $\left( \frac{R_0}{R} \right) = 1$, so that equation (20) reduces to

$$ \frac{R_0}{\rho U_0^2} \frac{dp}{dx} = \frac{16.6}{Re_0 U_0} - \frac{16}{Re_0}. $$

(30)

Integrating equation (30) between $x = -4l_0$ to $x = 4l_0$, the pressure drop in
The pressure drops have been calculated by the expressions (29) and (31) for stenosed and normal arteries using the data. These results have been shown in Fig. 8 along with Forrester and Young [12] for the purpose of comparison.

Fig. 8 depicts the variation of pressure drop with Reynolds numbers in the presence of stenosis. The velocities get elevated in comparison to Mishra and Kar’s model [8] of stenosis for unobstructed artery. These observations are in agreement with experimental observations of Forrester and young [12]. Hence slip velocity has a reducing effect on the pressure drop.

**TABLE – 1** Distributions of the non-dimensional pressure drop with time and slip velocity.

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<thead>
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<th>Time</th>
<th>$W/\overline{U_0}$</th>
<th>$X/L_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$-1/2$</td>
</tr>
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<td>With consideration of slip velocity (Present investigation)</td>
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<td>0</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
<td>Disregarding slip velocity (Forrester and Young [8])</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Mathematical Modeling of Unsteady Flow Through

Fig. 2: Variation of axial velocity for $t/T=0$, $Re_o=100$ with slip velocity.

Fig. 3: Variation of axial velocity for $t/T=3$, $Re_o=100$ with & without slip velocity.

Fig. 4: Variation of axial velocity for $t/T=0$, $Re_o=300$ with slip velocity.

Fig. 5: Variation of axial velocity for $t/T=3$, $Re_o=300$ with & without slip velocity.
References

Mathematical Modeling of Unsteady Flow Through
