p\text{th} order of Entire Harmonic Function

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Abstract:
In this paper we are reporting the \(p\text{th}\) lower order and \(p\text{th}\) lower type of an entire function \(H(r,\theta,\phi)\). These functions are obtained by various characterization in terms of \((\alpha_n)\) defined in [1] we also defined coefficient characterization of order and type of \(H(r,\theta,\phi)\)

Keywords: Entire harmonic function; order; type

1. Introduction:
If \(H(r,\theta,\phi)\) is a function and is harmonic in a neighborhood of origin in \(\mathbb{R}^3\).
\(H(r,\theta,\phi)\) has following expansion in spherical co-ordinate
\[
H(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( a_{nm}^{(1)} \cos m\phi + a_{nm}^{(2)} \sin n\phi \right) \sin^n \phi \cos^m \phi
\]
(1.1)

Where \(a_{nm}^{(1)}\) & \(a_{nm}^{(2)}\) are different coefficients.
This series converges absolutely and uniformly on a compact set of largest open ball centered at the origin which omits singularities of \(H(r,\theta,\phi)\).

Here \(x=rcos \theta \ y=rcos \theta \cos \phi \ z=rsin \theta \sin \phi\)

\(p_n^m(x)\) are associated Legendre function of first kind, \(n\text{th}\) degree of order \(m\) function was defined as
\[
p_n^m(x) = \left(1-x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} (p_n^m(x)).
\]
For \(H(r,\theta,\phi)\) entire define
\[ M(r) = M(r, H) = \max_{\theta, \phi} H(r, \theta, \phi) \]  

(1.2)

By [1] the \( p \)th order \( \rho^* \) and \( p \)th type \( T^* \) of \( H(r, \theta, \phi) \) are defined as

\[ \rho^* = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} \]  

(1.3)

\[ T^* = \limsup_{r \to \infty} \frac{\log \log M(r)}{\rho^*} \]  

(1.4)

For \( p = 1 \) the above definition same with classical definition of order and type.

Lower \( p \)th order \( \lambda^* \) and lower type \( \tau^* \) are defined by [3] as

\[ \lambda^* = \lambda^*(H) = \liminf_{r \to \infty} \frac{\log \log M(r)}{\log r} \]  

(1.5)

\[ \tau^* = \tau^*(H) = \liminf_{r \to \infty} \frac{\log \log M(r)}{\rho^*} \]  

(1.6)

For \( p = 2, 3, 4 \ldots \) where \( \log^0 x = x \) and \( \log^{[p]} = \log (\log^{[p-1]} x) \) we have consider \( p \)th lower order and \( p \)th lower type of harmonic function \( H(r, \theta, \phi) \) and obtain various characterization of these in terms of \( (\alpha_n) \).

Defined by [1] as

\[ \alpha_n = \max \left\{ \left( \frac{(n + m)!}{(m - n)!} \right)^{1/2} |a_m^n| \right\} \quad i = 1, 2. \]  

(1.7)

Also, by [1] defined function as

\[ f(z) = \sum_{n=0}^{\infty} \alpha_n \left( 1 + n^{-1/2} \right)^n z^n \]  

(1.8)

\[ g(z) = \sum_{n=0}^{\infty} \alpha_n \left( 1 + 2n \right)^{1/2} z^n \]

where \( \alpha_n \) defined above.

\[ H(r, \theta, \phi) \]

**Lemma 1:** If is entire Harmonic function the \( f(z) \) and \( g(z) \) are also entire function of complex variable \( z \) further

\[ 2^{1/2} m(r, f) \leq M(r) \leq 2 M(r, f) \]  

(1.9)

\[ m(r, g) = \max_n \left\{ \alpha_n \left( 1 + 2n \right)^{-1} \right\} \]

\[ M(r, f) = \max_{|z| < r} \left\| f(z) \right\| \]  

(1.10)

This result is obtained by Frayant [5, pp 27_28].
**Lemma 2:** Let \( f(z) \) and \( g(z) \) are entire functions defined as above then \( p^{th} \) order and \( p^{th} \) type of \( f(z) \) and \( g(z) \) are equal.

**PROOF** Let \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) be any entire function of \( p^{th} \) order \( \rho^*(F) \) and \( p^{th} \) type \( T^*(F) \)

Then it will be known by S. K. Bajpai G. P. Kapoor and O. P. Junenja [2]

\[
\rho^*(F) = \lim_{n \to \infty} \sup n \frac{\log \left( \frac{p-1}{p} \right) n}{\log(a_n)}
\]

\[
T^*(F) = \lim_{n \to \infty} \frac{\log \left( \frac{p-1}{p} \right) n}{e^2}
\]

Here for the function

\( f(z) = \sum_{n=0}^{\infty} a_n \left( 1 + n^{-1/2} \right)^{p} z^n \)

we have

\[
\frac{1}{\rho^*(f)} = \lim_{n \to \infty} \inf \frac{\log \left( \frac{p-1}{p} \right) n}{\log(a_n)}
\]

\[
= \lim_{n \to \infty} \frac{\log \alpha_n^{-1}}{n \log \left( \frac{p-1}{p} \right) n}
\]

Similarly for

\( g(z) = \sum_{n=0}^{\infty} a_n \left( 1 + 2n \right)^{-1/2} z^n \)

\[
\frac{1}{\rho^*(g)} = \lim_{n \to \infty} \inf \frac{\log \left( \frac{p-1}{p} \right) n}{\log(a_n)}
\]

\[
= \lim_{n \to \infty} \frac{\log \alpha_n^{-1} + 1/2 \log(1+2n)}{n \log \left( \frac{p-1}{p} \right) n}
\]

Hence \( \rho^*(f) = \rho^*(g) \) since \( f \) and \( g \) are of same order using (2. 2) we get

\( T^*(f) = T^*(g) \).

2. Result and Discussions
Theorem -1 let $H(r, \theta, \phi)$ be an entire Harmonic function of $p^{th}$ order $\rho^*$ and $p^{th}$ lower order $\lambda^*$ and $p^{th}$ type $T^*$ also lower $p^{th}$ type $\tau^*$ if $f(z)$ and $g(z)$ are entire functions defined above then
\[ \rho^*(f) = \rho^*(g) = \rho^* \]
\[ T^* (f) = T^* (g) = T^* \]
\[ \lambda^* (g) \leq \lambda^* \leq \lambda^* (f) \]
\[ \tau^* (g) \leq \tau^* \leq \tau^* (f) \]

Proof : By Srivastava’s study[1]
\[ 2^{-1} m(r, g) \leq M(r) \leq 2 M(r, g) \]

We have,
\[ \lim \sup_{r \to \infty} \inf \frac{\log^p M(r, g)}{\log r} \leq \lim \sup_{r \to \infty} \inf \frac{\log^p M(r)}{\log r} \leq \lim \sup_{r \to \infty} \inf \frac{\log^p M(r, f)}{\log r} \]

\[ \log^p M(r, F) \equiv \log^p m(r, f) \] as $r \to \infty$ result is defined in [6]

Hence from above we get
\[ \rho^* (g) \leq \rho^* \leq \rho^* (f) ; \lambda^* (g) \leq \lambda^* \leq \lambda^* (f) \]

since $\rho^* (g) = \rho^* (g)$

Thus we obtained (2. 1) and (2. 3) from (2. 5)
\[ \lim \sup_{r \to \infty} \frac{\log^{p-1} m(r, g)}{r^\rho^*} \leq \lim \sup_{r \to \infty} \frac{\log^{p-1} M(r)}{r^\rho^*} \leq \lim \sup_{r \to \infty} \frac{\log^{p-1} M(r, f)}{r^\rho^*} \]

Hence from lemma (2) we have (2. 2) and (2. 4).

Theorem 2: Let $H(r, \theta, \phi)$ be an entire function of order $\rho^*$ and lower order $\lambda^*$ and lower type $\tau^*$. If \( \left( \frac{a_n}{a_{n+1}} \right) \) is non decreasing function for $n \geq n_0$ then
\[ \lambda^* = \lim_{r \to \infty} \sup_{n \to \infty} \frac{n \log^p n}{\log (a_n)} \]

Proof: For entire function
\[ F(z) = \sum_{n=0}^{\infty} a_n z^n \quad |a_n| \text{non decreasing function of } n \text{ for } n \geq n_0 \text{ then} \]
\[ \lambda^* (F) = \lim_{n \to \infty} \sup_{n \to \infty} \frac{n \log^p n}{\log (a_n)} \] If \( \left( \frac{a_n}{a_{n+1}} \right) \) be a non decreasing function of $n$
then

\[ \lambda^*(f) = \limsup_{n \to \infty} \frac{n \log n}{\log \left( \alpha_n \right)^{-1} - n \log \left( 1 + \frac{1}{n} \right)} \]

= \liminf_{n \to \infty} \frac{n \log n}{\log \left( \alpha_n \right)^{-1}}

Similarly for \( g(z) = \sum_{n=0}^{\infty} \alpha_n (1 + 2n)^{-1/2} z^n \)

\[ \lambda^*(g) = \liminf_{n \to \infty} \frac{n \log(n) n}{\log(\alpha_n)^{-1} - \frac{1}{2} \log(1 + 2n)} = \liminf_{n \to \infty} \left\{ \frac{n \log(n) n}{\log(\alpha_n)^{-1}} \right\} \]

From (2.1) we have \( \lambda^* = \lim_{n \to \infty} \frac{n \log(n) n}{\log(\alpha_n)^{-1}} \).

**Theorem 3:** Let \( H(r, \theta, \phi) \) be an entire Harmonic function lower \( p^\text{th} \) order \( \lambda^* \) and \( \left( \frac{\alpha_n}{\alpha_{n+1}} \right) \) is non decreasing function for \( n > n_0 \) then

\[ \lambda^* = \lim_{n \to \infty} \frac{\log(n) n}{\log \left( \frac{\alpha_n}{\alpha_{n+1}} \right)} \]  \hspace{1cm} (2.7)

**Proof:** For an entire function

\[ F(z) = \sum_{n=0}^{\infty} a_n z^n \]

defined by [2]

\[ \lambda^*(F) = \liminf_{n \to \infty} \frac{\log(n) n}{\log \left( \frac{a_n}{a_{n+1}} \right)} \]

Provided \( \left| \frac{a_n}{a_{n+1}} \right| \) non decreasing function of \( n \) for \( n < n_0 \)

Using the condition on \( [\alpha_n] \) we can easily show as in above theorem

\[ \lambda^*(f) = \liminf_{n \to \infty} \frac{\log(n) n}{\log \left( \frac{\alpha_n}{\alpha_{n+1}} \right)} \]
Applying to \( g(z) = \sum_{n=0}^{\infty} (1 + 2n)^{-1/2} \alpha_n z^n \)

We have

\[
\lambda^*(f) = \lim_{n \to \infty} \frac{\log^{|\rho|} n}{\log(\frac{\alpha_n}{\alpha_{n+1}})} + \frac{1}{2} \log(\frac{1 + 2(n + 1)}{1 + 2n})
\]

\[
= \lim_{n \to \infty} \frac{\log^{|\rho|} n}{\log(\frac{\alpha_n}{\alpha_{n+1}})}
\]

Thus we relation by using (2.3)

ACKNOWLEDGEMENTS
The authors are thankful to the referee for his helpful comments and suggestions

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