A new fractional derivative on time scales

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**Abstract**

In this paper, we define and study the delta fractional derivative on time scales. Many basic properties of delta fractional derivative will be obtained.

**AMS subject classification:** 26A33, 26E70.

**Keywords:** Fractional calculus, delta fractional derivative, time scales.

1. Introduction

Fractional Calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians. Nowadays, the fractional calculus attracts many scientists and engineers. There are several applications of this mathematical phenomenon in mechanics, physics, chemistry, control theory and so on [1-8].

A time scale is any nonempty closed subset of the real line. The theory of time scales is a fairly new area of research. It was introduced in Stefan Hilger’s 1988 Ph.D. thesis [9] to unify the theory of difference equations and the theory of differential equations. It has been extensively studied on various aspects by several authors [10-15]. The idea to join the two–subjects the fractional calculus and the calculus on time scales–and to develop a Fractional Calculus on time scales, was born with the Ph.D. thesis of Bastos [16]. Recently, a nabla, a delta, and a symmetric fractional calculus on arbitrary nonempty closed subsets of the real numbers was introduced and developed in [17, 18]. Especially, in [19], Nadia Benkhettou, Salima Hassani and Delfim F.M. Torres introduced a conformable time-scale fractional calculus, which providing a natural extension of the conformable fractional calculus. In this paper, we introduce and investigate a new concept of delta fractional derivative on time scales but is different from [17, 18].

\(^1\)Foundation item: Supported by Educational Commission of Hubei Province of China (Q20152505).
2. Definitions and Basic Properties

A time scale \( \mathbb{T} \) is a nonempty closed subset of real numbers \( \mathbb{R} \) with the subspace topology inherited from the standard topology of \( \mathbb{R} \). For \( a, b \in \mathbb{T} \) we define the closed interval \([a, b]_\mathbb{T}\) by \([a, b] = \{ t \in \mathbb{T} : a \leq t \leq b \}\). For \( t \in \mathbb{T} \) we define the forward jump operator \( \sigma(t) \) by \( \sigma(t) = \inf \{ s > t : s \in \mathbb{T} \} \) where \( \inf \emptyset = \sup \mathbb{T} \), while the backward jump operator \( \rho(t) \) is defined by \( \rho(t) = \sup \{ s < t : s \in \mathbb{T} \} \) where \( \sup \emptyset = \inf \mathbb{T} \).

If \( \sigma(t) > t \), we say that \( t \) is right-scattered, while if \( \rho(t) < t \), we say that \( t \) is left-scattered. If \( \sigma(t) = t \), we say that \( t \) is right-dense, while if \( \rho(t) = t \), we say that \( t \) is left-dense. A point \( t \in \mathbb{T} \) is dense if it is right and left dense; isolated if it is right and left scattered. The forward graininess function \( \mu(t) \) and the backward graininess function \( \eta(t) \) are defined by \( \mu(t) = \sigma(t) - t \), \( \eta(t) = t - \rho(t) \) for all \( t \in \mathbb{T} \) respectively.

If \( \sup \mathbb{T} \) is finite and right-scattered, then we define \( \mathbb{T}^k := \mathbb{T} \setminus \sup \mathbb{T} \), otherwise \( \mathbb{T}^k := \mathbb{T} \); if \( \inf \mathbb{T} \) is finite and right-scattered, then \( \mathbb{T}_k := \mathbb{T} \setminus \inf \mathbb{T} \), otherwise \( \mathbb{T}_k := \mathbb{T} \). We set \( \mathbb{T}_k^k := \bigcap \mathbb{T}_k \).

A function \( f : [a, b]_\mathbb{T} \to \mathbb{R} \) is called regulated provided its right-sided limits exist at all right-dense point of \([a, b]_\mathbb{T}\) and its left-sided limits exist at all left-dense point of \((a, b)_{\mathbb{T}}\).

A function \( f : \mathbb{T} \to \mathbb{R} \) is delta differentiable at \( t \in \mathbb{T}^k \) if there exists a number \( f^{\Delta}(t) \) such that, for each \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that

\[
|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|
\]

for all \( s \in U \). We call \( f^{\Delta}(t) \) the delta derivative of \( f \) at \( t \) and we say that \( f \) is delta differentiable if \( f \) is delta differentiable for all \( t \in \mathbb{T}^k \).

Throughout this paper, \( \alpha \in (0, 1] \).

**Definition 2.1.** Let \( \mathbb{T} \) be a time scale and \( \alpha \in (0, 1] \). A function \( f : \mathbb{T} \to \mathbb{R} \) is delta fractional differentiable of order \( \alpha \) at \( t \in \mathbb{T}^k \) if there exists a number \( T_\alpha(f^{\Delta})(t) \) such that, for each \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that

\[
|f^\alpha(t) - f(s) - T_\alpha(f^{\Delta})(t)(\sigma(t)^\alpha - s^\alpha)| \leq \varepsilon|\sigma(t)^\alpha - s^\alpha|
\]

for all \( s \in U \). We call \( T_\alpha(f^{\Delta})(t) \) the delta fractional derivative of \( f \) of order \( \alpha \) at \( t \) and we say that \( f \) is delta fractional differentiable if \( f \) is delta fractional differentiable for all \( t \in \mathbb{T}^k \).

Some useful properties of the delta fractional derivative of \( f \) are given in the next theorem.

**Theorem 2.2.** Let \( \mathbb{T} \) be a time scale, \( t \in \mathbb{T}^k \) and \( \alpha \in (0, 1] \). Then we have the following:

1. If \( f \) is delta fractional differentiable of order \( \alpha \) at \( t \), then \( f \) is continuous at \( t \).

2. If \( f \) is continuous at \( t \) and \( t \) is right-scattered, then \( f \) is delta fractional differentiable of order \( \alpha \) at \( t \) with

\[
T_\alpha(f^{\Delta})(t) = \frac{f^\alpha(t) - f(t)}{\sigma(t)^\alpha - t^\alpha}.
\]
(3) If \( t \) is right-dense, then \( f \) is delta fractional differentiable of order \( \alpha \) at \( t \) if and only if the limit
\[
\lim_{s \to t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha}
\]
exists as a finite number. In this case,
\[
T_\alpha(f/f^\Delta_1)(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha}.
\]

(4) If \( f \) is delta fractional differentiable of order \( \alpha \) at \( t \), then
\[
f^\sigma(t) = f(t) + T_\alpha(f/f^\Delta_1)(t)(\sigma(t)^\alpha - t^\alpha).
\]

**Proof.** Part (1). Assume that \( f \) is delta fractional differentiable at \( t \), then for each \( \epsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that
\[
|f^\sigma(t) - f(s) - T_\alpha(f/f^\Delta_1)(t)(\sigma(t)^\alpha - s^\alpha)| \leq \epsilon^*|\sigma(t)^\alpha - s^\alpha|
\]
for all \( s \in U \), here
\[
\epsilon^* = \frac{\epsilon}{2(1 + |t|^\alpha + |T_\alpha(f/f^\Delta_1)(t)|)}.
\]

Then for all \( s \in U \cap (t - \epsilon^*, t + \epsilon^*) \), we have that
\[
|f^\sigma(t) - f(s)| = |f^\sigma(t) - f(s) - T_\alpha(f/f^\Delta_1)(t)(\sigma(t)^\alpha - s^\alpha) - T_\alpha(f/f^\Delta_1)(t)(\sigma(t)^\alpha - t^\alpha) + T_\alpha(f/f^\Delta_1)(t)(t^\alpha - s^\alpha)|
\]
\[
\leq \epsilon^*|\sigma(t)^\alpha - s^\alpha| + \epsilon^*|\sigma(t)^\alpha - t^\alpha| + |T_\alpha(f/f^\Delta_1)(t)(t^\alpha - s^\alpha)|
\]
\[
\leq 2\epsilon^* (1 + |t|^\alpha + |T_\alpha(f/f^\Delta_1)(t)|)
\]
\[
= \epsilon.
\]
It follows that \( f \) is continuous at \( t \).

Part (2). Assume that \( f \) is continuous at \( t \) and \( t \) is right-scattered, By continuity,
\[
\lim_{s \to t} \frac{f^\sigma(t) - f(s)}{\sigma(t)^\alpha - s^\alpha} = \frac{f^\sigma(t) - f(t)}{\sigma(t)^\alpha - t^\alpha}.
\]
Hence, given \( \epsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that
\[
\left| \frac{f^\sigma(t) - f(s)}{\sigma(t)^\alpha - s^\alpha} - \frac{f^\sigma(t) - f(t)}{\sigma(t)^\alpha - t^\alpha} \right| \leq \epsilon
\]
for all \( s \in U \). It follows that
\[
\left| f^\sigma(t) - f(s) - \frac{f^\sigma(t) - f(t)}{\sigma(t)^\alpha - t^\alpha}(\sigma(t)^\alpha - s^\alpha) \right| \leq \epsilon|\sigma(t)^\alpha - s^\alpha|.
for all $s \in U$. Hence we get the desired result

$$T_\alpha(f^\Delta)(t) = \frac{f^\Delta(t) - f(t)}{\sigma(t)^\alpha - t^\alpha}.$$  

Part (3). Assume that $f$ is delta fractional differentiable at $t$ and $t$ is right-dense. Then for each $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|f^\alpha(t) - f(s) - T_\alpha(f^\Delta)(t)(\sigma(t)^\alpha - s^\alpha)| \leq \varepsilon|\sigma(t)^\alpha - s^\alpha|$$

for all $s \in U$. Since $\sigma(t) = t$ we have that

$$|f(t) - f(s) - T_\alpha(f^\Delta)(t)(t^\alpha - s^\alpha)| \leq \varepsilon|t^\alpha - s^\alpha|$$

for all $s \in U$. It follows that

$$\left| \frac{f(t) - f(s)}{t^\alpha - s^\alpha} - T_\alpha(f^\Delta)(t) \right| \leq \varepsilon$$

for all $s \in U$, $s \neq t$. Hence we get the desired result

$$T_\alpha(f^\Delta)(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha}.$$  

On the other hand, if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha}$$

exists as a finite number and is equal to $J$, then for each $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|f(t) - f(s) - J(t^\alpha - s^\alpha)| \leq \varepsilon|t^\alpha - s^\alpha|$$

for all $s \in U$. Since $t$ is right-dense, we have that

$$|f^\alpha(t) - f(s) - J(\sigma(t)^\alpha - s^\alpha)| \leq \varepsilon|\sigma(t)^\alpha - s^\alpha|.$$  

Hence, $f$ is delta fractional differentiable at $t$ and

$$T_\alpha(f^\Delta)(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha}.$$  

Part (4). If $t$ is right-dense, then $\sigma(t) = t$ and we have that

$$f^\alpha(t) = f(t) + T_\alpha(f^\Delta)(t)(\sigma(t)^\alpha - t^\alpha).$$

If $t$ is right-scattered, then $\sigma(t) > t$, then by (2)

$$f^\alpha(t) = f(t) + T_\alpha(f^\Delta)(t)(\sigma(t)^\alpha - t^\alpha).$$

\[\blacksquare\]

Corollary 2.3. Again we consider the two cases $T = \mathbb{R}$ and $T = \mathbb{Z}$.  

(1) If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta fractional differentiable at $t \in \mathbb{R}$ if and only if the limit
\[ \lim_{s \to t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha} \]
equalsas a finite number. In this case,
\[ T_\alpha(f^\Delta)(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha} . \]
If $\alpha = 1$, then we have that
\[ T_\alpha(f^\Delta)(t) = f^\Delta(t) = f'(t) . \]

(2) If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is delta fractional differentiable at $t \in \mathbb{Z}$ with
\[ T_\alpha(f^\Delta)(t) = \frac{f(t+1) - f(t)}{(t+1)^\alpha - t^\alpha} \]
If $\alpha = 1$, then we have that
\[ T_\alpha(f^\Delta)(t) = f(t+1) - f(t) = \Delta f(t) , \]
where $\Delta$ is the usual forward difference operator.

**Example 2.4.** If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = C$ for all $t \in \mathbb{T}$, where $C \in \mathbb{R}$ is constant, then
\[ T_\alpha(f^\Delta)(t) \equiv 0 . \]
This is because for any $\epsilon > 0$,
\[ |f^\sigma(t) - f(s) - 0 \cdot (\sigma(t)^\alpha - s^\alpha)| = |C - C| = 0 \leq \epsilon |(\sigma(t)^\alpha - s^\alpha)| \]
holda for all $s \in \mathbb{T}$. 

**Theorem 2.5.** Assume $f, g : \mathbb{T} \to \mathbb{R}$ are delta fractional differentiable at $t \in \mathbb{T}^k$. Then:

(1) for any constant $\lambda_1, \lambda_2$, the sum $\lambda_1 f + \lambda_2 g : \mathbb{T} \to \mathbb{R}$ is delta fractional differentiable at $t \in \mathbb{T}^k$ with
\[ T_\alpha((\lambda_1 f + \lambda_2 g)^\Delta)(t) = \lambda_1 T_\alpha(f^\Delta)(t) + \lambda_2 T_\alpha(g^\Delta)(t) . \]

(2) If $f$ and $g$ are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is delta fractional differentiable at $t$ with
\[ T_\alpha((fg)^\Delta)(t) = T_\alpha(f^\Delta)(t)g(t) + f^\sigma(t)T_\alpha(g^\Delta)(t) \]
\[ = f(t)T_\alpha(g^\Delta)(t) + T_\alpha(f^\Delta)(t)g^\sigma(t) . \]
(3) If \( f(t)^\sigma(t) \neq 0 \), then \( \frac{1}{f} \) is delta fractional differentiable at \( t \) with
\[
T_\alpha \left( \frac{1}{f} \right)^\Delta(t) = -\frac{T_\alpha(f^\Delta)(t)}{f(t)^\sigma(t)}.
\]

(4) If \( g(t)^\sigma(t) \neq 0 \), then \( \frac{f}{g} \) is delta fractional differentiable at \( t \) with
\[
T_\alpha \left( \frac{f}{g} \right)^\Delta(t) = \frac{T_\alpha(f^\Delta)(t)g(t) - f(t)T_\alpha(g^\Delta)(t)}{g(t)^\sigma(t)}.
\]

Proof. Part (1). Let \( \varepsilon > 0 \). Then there exists neighborhoods \( U_1 \) and \( U_2 \) of \( t \) with
\[
|\lambda_1 f^\sigma(t) - \lambda_1 f(s) - \lambda_1 T_\alpha(f^\Delta)(t)(\sigma(t)^\alpha - s^\alpha)| \leq \frac{\varepsilon}{2}|\lambda_1(\sigma(t)^\alpha - s^\alpha)|
\]
for all \( s \in U_1 \) and
\[
|\lambda_2 g^\sigma(t) - \lambda_2 g(s) - \lambda_2 T_\alpha(g^\Delta)(t)(\sigma(t)^\alpha - s^\alpha)| \leq \frac{\varepsilon}{2}|\lambda_2(\sigma(t)^\alpha - s^\alpha)|
\]
for all \( s \in U_2 \).

Let \( U = U_1 \cap U_2 \), \( \lambda = \max\{\lambda_1, \lambda_2\} \). Then we have for all \( s \in U \)
\[
|((\lambda_1 f^\sigma + \lambda_2 g^\sigma)(t) - (\lambda_1 f + \lambda_2 g)(s)) - (\lambda_1 T_\alpha(f^\Delta)(t) + \lambda_2 T_\alpha(g^\Delta)(t))((\sigma(t)^\alpha - s^\alpha)| \leq \frac{\varepsilon}{2}|\lambda(\sigma(t)^\alpha - s^\alpha)|.
\]

Therefore \( \lambda_1 f + \lambda_2 g \) is delta fractional differentiable at \( t \in \mathbb{T}^k \) with
\[
T_\alpha((\lambda_1 f + \lambda_2 g)^\Delta)(t) = \lambda_1 T_\alpha(f^\Delta)(t) + \lambda_2 T_\alpha(g^\Delta)(t).
\]

Part (2). Let \( 0 < \varepsilon < 1 \). Define
\[
\varepsilon^* = \frac{\varepsilon}{1 + |g^\sigma(t)| + |f(t)| + |T_\alpha(g^\Delta)(t)|},
\]
then \( 0 < \varepsilon^* < 1 \). \( f, g : \mathbb{T} \rightarrow \mathbb{R} \) are delta fractional differentiable at \( t \in \mathbb{T}^k \). Then there exists neighborhoods \( U_1 \) and \( U_2 \) of \( t \) with
\[
|f^\sigma(t) - f(s) - T_\alpha(f^\Delta)(t)(\sigma(t)^\alpha - s^\alpha)| \leq \varepsilon^*|\sigma(t)^\alpha - s^\alpha|.
\]
for all \( s \in U_1 \) and
\[
|g^\sigma(t) - g(s) - T_\alpha(g^\Delta)(t)(\sigma(t)^\alpha - s^\alpha)| \leq \epsilon^*|\sigma(t)^\alpha - s^\alpha|
\]
for all \( s \in U_2 \).

From Theorem 2.2 (1), there exists neighborhoods \( U_3 \) of \( t \) with
\[
|f(t) - f(s)| \leq \epsilon^*
\]
for all \( s \in U_3 \).

Let \( U = U_1 \cap U_2 \cap U_3 \). Then we have for all \( s \in U \)
\[
|[f^\sigma(t)g^\sigma(t) - f(s)g(s)] - [T_\alpha(f^\Delta)(t)g^\sigma(t) + f(t)T_\alpha(g^\Delta)(t)](\sigma(t)^\alpha - s^\alpha)|
\leq
\]
\[
||g^\sigma(t) - g(s)| - T_\alpha(g^\Delta)(t)(\sigma(t)^\alpha - s^\alpha)|g^\sigma(t)|
+||f(t) - f(s) - T_\alpha(f^\Delta)(t)(\sigma(t)^\alpha - s^\alpha)||f(t)|
\leq
\]
\[
\epsilon^*|\sigma(t)^\alpha - s^\alpha| \cdot (|g^\sigma(t)| + |f(t)| + \epsilon^* + |T_\alpha(g^\Delta)(t)|).
\]
Thus
\[
T_\alpha(fg)^\Delta(t) = f(t)T_\alpha(g^\Delta)(t) + T_\alpha(f^\Delta)(t)g^\sigma(t).
\]
The other product rule formula follows by interchanging the role of functions \( f \) and \( g \).

**Part (3).** From Example 2.4 we have that
\[
T_\alpha \left( f \cdot \frac{1}{f} \right)^\Delta(t) = T_\alpha(1)^\Delta(t) = 0.
\]
Therefore,
\[
T_\alpha \left( \frac{1}{f} \right)^\Delta(t) f^\sigma(t) + T_\alpha(f^\Delta)(t) \frac{1}{f(t)} = 0
\]
and consequently
\[
T_\alpha \left( \frac{1}{f} \right)^\Delta(t) = -\frac{T_\alpha(f^\Delta)(t)}{f(t)f^\sigma(t)}.
\]

**Part (4).** We use (2) and (3) to calculate
\[
T_\alpha \left( \frac{f}{g} \right)^\Delta(t) = f(t)T_\alpha \left( \frac{1}{g} \right)^\Delta(t) + T_\alpha(f^\Delta)(t) \frac{1}{g^\sigma(t)}
\]
\[
= -f(t)T_\alpha \left( \frac{g^\Delta}{g(t)g^\sigma(t)} \right) + T_\alpha(f^\Delta)(t) \frac{1}{g^\sigma(t)}
\]
\[
= \frac{T_\alpha(f^\Delta)(t)g(t) - f(t)T_\alpha(g^\Delta)(t)}{g(t)g^\sigma(t)}.
\]

\[\blacksquare\]
Acknowledgements

The authors are grateful to the referee for his or her careful reading of the manuscript and for valuable and helpful suggestions.

References


