

# On the Embedding of $\Gamma$ -semigroup Amalgam using Partial $\Gamma$ -Groupoid

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## Abstract

A partial groupoid consists of a set equipped with a partial binary operation. Analogously we defined a partial  $\Gamma$ -groupoid and  $\Gamma$ -semigroup amalgam. In this paper we discuss embeddability of a  $\Gamma$ -semigroup amalgam using partial  $\Gamma$ -groupoids.

**Keywords:**  $\Gamma$ -semigroups , Semigroup Amalgam ,  $\Gamma$ -monomorphisms ,  $\Gamma$ -groupoid

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## 1. INTRODUCTION

The embeddability of group amalgam was positively done by Schreier [7] in 1927. An alternative proof of Schreier's theorem was given by R. Baer [2] - [4] using partial groupoids. N. Kimura [13] showed in 1957 that not all semigroup amalgams are embeddable. A detail study of semigroup amalgams were done by J.M. Howie [10]. Howie [11] in 1962 proved that the amalgam is embeddable if  $U$  is a unitary subsemigroup. Later new approaches regarding the amalgamation using unitary subsemigroups were brought by G.B.Preston [16], T.E. Hall[8], G.Lallement[14] and D.Dekov [6].

The concept of  $\Gamma$ -semigroups was introduced by M K Sen [18]. Let  $M$  and  $\Gamma$  be two non-empty sets, then  $M$  is called a  $\Gamma$ -semigroup if the following axioms are satisfied : for all  $a, b, c \in M$  and  $\alpha, \beta \in \Gamma$

- $a\alpha b \in M$ .
- $(a\alpha b)\beta c = a\alpha(b\beta c)$ .

Every semigroup is a  $\Gamma$ -semigroup and for any fixed  $\alpha$  in  $\Gamma$  every  $\Gamma$ -semigroup is a semigroup  $(M, \cdot)$  with operation  $a \cdot b = a\alpha b$ .

Let  $M$  be a  $\Gamma$ -semigroup,  $M_1$  and  $M_2$  be two non-empty subsets of  $M$ , then the set  $\{a\gamma b : a \in M_1, b \in M_2, \gamma \in \Gamma\}$  is  $M_1\Gamma M_2$ . A non-empty subset  $M_1 \subset M$  of a  $\Gamma$ -semigroup is called a  $\Gamma$ -subsemigroup of  $M$  if  $M_1\Gamma M_1 \subset M_1$ . For  $M, N$  be two  $\Gamma$ -semigroups, a mapping  $f : M \rightarrow N$  satisfying  $f(a\gamma b) = f(a)\gamma f(b)$  for each  $a, b \in M$  and  $\gamma \in \Gamma$  is called a  $\Gamma$ -semigroup homomorphism. In addition if  $f(a_1\gamma b_1) = f(a_2\gamma b_2)$  implies  $a_1\gamma b_1 = a_2\gamma b_2$  then the mapping is called  $\Gamma$ -monomorphism.

An amalgam of two semigroups  $S_1$  and  $S_2$ ,  $\mathcal{A} = [U, S_1, S_2, \phi_1, \phi_2]$  where  $U = S_1 \cap S_2$  is a subsemigroup of both  $S_1$  and  $S_2$  and  $\phi_i : U \rightarrow S_i$  for  $i = 1, 2$  are monomorphisms. By a partial  $\Gamma$ -groupoid we mean a set with partial binary operation and it can be seen that ([5], section 9.4) every semigroup amalgam determines a partial groupoid and if that partial groupoid is embeddable in a semigroup then the corresponding amalgam is also embeddable. The natural question in this context is, when can the partial groupoid of a given semigroup amalgam be embeddable? Answer to this question was positively done by Howie in [11].

In a group amalgam, its free product amalgamating the core has the property that equivalent reduced words have same length. In the case of semigroup amalgams too this property holds by Dekov [6]. That is, for semigroup amalgams embeddable in a group and such that in the free products amalgamating the common subsemigroups, equivalent reduced words have the same length. In this paper we extend this result to the case of  $\Gamma$ -semigroup amalgams.

## 2. PARTIAL $\Gamma$ -GROUPOID

A *partial  $\Gamma$ -groupoid* is a tuple  $M = (M, \Gamma, D, \mu)$ , where  $M$  is a non-empty set,  $D \subset M \times \Gamma \times M$  and  $\mu : D \rightarrow M$  is the mapping, we write

$$\mu(x, \gamma, y) = x\gamma y$$

$$(x, \gamma, y)_D \text{ if and only if } (x, \gamma, y) \in D.$$

Let  $M = (M, \Gamma, D, \mu)$  be a partial  $\Gamma$ -groupoid and  $F(M)$  be the free  $\Gamma$ -semigroup on the set  $M$ . The element  $(x_1, \gamma_1, x_2, \gamma_2, \dots, x_n) \in F(M)$  is called a word of length  $n$ . The word  $(x_1, \gamma_1, x_2, \gamma_2, \dots, x_n)$  is said to be reduced if for  $i = 1, \dots, n - 1$ ,  $(x_i, \gamma_i, x_{i+1}) \notin D$ . We write

$$(x_1, \gamma_1, x_2, \gamma_2, \dots, x_n)_D \text{ if and only if } (x_i, \gamma_i, x_{i+1})_D \text{ for } i = 1, \dots, n - 1.$$

Let  $\rho_0$  be a relation on  $F(M)$  defined by  $(x\gamma y, (x, \gamma, y)) \in \rho_0$  if and only if  $(x, \gamma, y)_D$ . Let  $\rho$  be the congruence of  $F(M)$  generated by  $\rho_0$ . Then  $U(M) = F(M)/\rho$  is the universal  $\Gamma$ -semigroup of the partial  $\Gamma$ -groupoid  $M$ . Two words are equivalent if they belongs to the same congruence class of  $\rho$ . That is,  $X$  and  $Y$  are equivalent if  $X \sim Y \pmod{\rho}$ .

**Definition 2.1** The word  $v = (v_1, \gamma_1, v_2, \gamma_2, \dots, \gamma_{n-1}, v_n)$  in  $F(M)$  is said to be 1-step contractable if there exists an integer  $i$  such that  $1 < i < n$  and the element  $(v_i, \gamma_i, v_{i+1}) \in D$ .

The word  $v$  of length  $n$  is said to be contractable if  $1 < n$  and if there exist a word  $w$  of length  $n - 1$  which is directly equivalent to  $v$ .

**Definition 2.2** If  $v(1), \gamma_1, v(2), \gamma_2, \dots, \gamma_{m-1}, v(m)$  are contractions of a word  $v$  and if  $v(i)$  and  $v(i + 1)$  possess, for  $0 < i < m$ , a common contraction then  $v(1)$  and  $v(m)$  are connected contractions of  $v$ .

**Definition 2.3** A partial  $\Gamma$ -groupoid  $M = (M, \Gamma, D, \mu)$  is said to be a partial  $\Gamma$ -semigroup if it is embeddable in a  $\Gamma$ -semigroup.

**Theorem 2.4** A partial  $\Gamma$ -groupoid  $M$  is a partial  $\Gamma$ -semigroup if and only if  $x \sim y \pmod{\rho}$  implies  $x = y$ , for all  $x, y \in G$ .

**Definition 2.5** A partial  $\Gamma$ -semigroup  $M = (M, \Gamma, D, \mu)$  is said to be pre- $\Gamma$ -semigroup if it satisfies the property: equivalent reduced words have the same length.

We propose the following conditions on partial  $\Gamma$ -groupoid so that it will be embeddable in a  $\Gamma$ -semigroup:

(M1) For all  $x, y, z \in M$  and  $\alpha, \beta \in \Gamma$  if  $(x, \alpha, y\beta z)_D, (x\alpha y, \beta, z)_D$  then  $x\alpha(y\beta z) = (x\alpha y)\beta z$ .

- (M2) For all  $x, y, z \in M$  and  $\alpha, \beta \in \Gamma$  if  $(x, \alpha, y\beta z)_D$  if and only if  $(x\alpha y, \beta, z)_D$ .
- (M3) For all  $w, x, y, z \in M$  and  $\alpha, \beta, \gamma \in \Gamma$  if  $(w, \alpha, x, \beta, y, \gamma, z)_D$  then either  $(w, \alpha, x\beta y)_D$  or  $(x\beta y, \gamma, z)_D$ .
- (M4) For all  $w, x, y, z \in M$  and  $\alpha, \beta, \gamma \in \Gamma$  if  $(x, \alpha, y\beta z)_D$  and  $(w, \gamma, x\alpha y)_D$  then either  $(w, \gamma, x)_D$  or  $(x\alpha y, \beta, z)_D$ .
- (M5) For all  $w, x, y, z \in M$  and  $\alpha, \beta, \gamma \in \Gamma$  if  $(w, \gamma, x, \alpha, y)_D$  and  $(x\alpha y, \beta, z)_D$  the either  $(w, \gamma, x\alpha y)_D$  of  $(y, \beta, z)_D$ .

**Theorem 2.6** *A partial  $\Gamma$ -groupoid  $M = (M, \Gamma, D, \mu)$  is a pre- $\Gamma$ -semigroup if and only if it satisfies axioms (M1) to (M5).*

**Proof 2.7** *Let  $M = (M, \Gamma, D, \mu)$  be a partial  $\Gamma$ -groupoid satisfy axioms (M1) to (M5). Now it is enough to show that, if the word  $X$  has two irreducible reductions then they have the same length. For, we proceed by induction on the length of the word.*

*If  $X$  is a word of length 1, then it is trivially true. Suppose  $1 < n$  and assume the condition is true for all words whose length is less than  $n$ . Let  $X$  be a word of length  $n$ , if  $X$  is irreducible then our contention is again trivially true. Finally suppose that  $X$  is reducible and  $x'$  and  $x''$  are irreducible reductions of  $X$ , consequently there exists contractions  $v'$  and  $v''$  such that  $x'$  is a reduction of  $v'$  and  $x''$  is a contraction of  $v''$ , if these two contractions are both irreducible, then  $x' = v'$  and  $x'' = v''$  are both of length  $n - 1$ . If  $v'$  and  $v''$  are not both irreducible, then they are connected, consequently there exists contractions  $v(0), v(1), \dots, v(k)$  of  $X$  such that  $v' = v(0)$ ,  $v'' = v(k)$  and such that  $0 \leq i < k$ ,  $v(i)$  and  $v(i + 1)$  possess for a common contraction  $u(i)$ . Since  $v(i)$  has length  $n - 1$ , it follows from the inductive hypothesis that all the irreducible reductions of  $v(i)$  has the same length  $m(i)$ . But  $u(i)$  has an irreducible reduction which is necessarily a reduction of both  $v(i)$  and of  $v(i + 1)$ . Consequently  $m(i) = m(i + 1)$  for all  $i$ , so that  $m(0) = m(k)$ . But  $m(0)$  is the length of  $x'$  and  $m(k)$  is the length of  $x''$ . Hence  $x'$  and  $x''$  have the same length.*

*Now on the other hand assume that similar reduced words have same length is satisfied. Then if  $(x, \alpha, y)_D$ ,  $(y, \beta, z)_D$  and  $(x\alpha y, \beta, z)_D$ , then we have*

$$(x\alpha y)\beta z \sim (x\alpha y, \beta, z)_D \sim (x, \alpha, y, \beta, z)_D \sim (x, \alpha, y\beta z)_D$$

*So,  $(x, \alpha, y\beta z)_D$  is reducible. Hence  $x\alpha(y\beta z)$ , consequently we have  $(x\alpha y)\beta z = x\alpha(y\beta z)$ .*

Similarly we can show the existence of  $(x\alpha y, \beta, z)$  and  $(x\alpha y)\beta z = x\alpha(y\beta z)$ . Hence (M1) holds. If the elements  $(x, \alpha, y)_D$ ,  $(y, \beta, z)_D$  and  $(c, \gamma, d)_D$  then we have

$$(a\alpha b, \beta, c\gamma d)_D \sim (a, \alpha, b, \beta, c, \gamma, d)_D \sim (a, \alpha, b\beta c, \gamma, d)$$

then it follows that the last of these words is reducible and hence the existence of the words  $(a, \alpha, b\beta c)_D$  and  $(b\beta c, \gamma, d)_D$ .

Also the products  $a\alpha b, b\beta c$  and  $(b\beta c, \gamma, d)_D$  then we obtain the word similarities

$$(a, \alpha, b\beta c, \gamma, d)_D \sim (a, \alpha, b, \beta, c, \gamma, d)_D \sim (a\alpha b, \beta, c, \gamma, d)_D$$

As before, we can deduce from the reducibility of the last of these words and therefore the existence of atleast one of the products  $(a\alpha b, \beta, c)_D$  and  $(c, \gamma, d)_D$  in  $F(M)$  so that the validity of (M2) is established.

### 3. QUASI $\Gamma$ -UNITARY $\Gamma$ -SUBSEMIGROUPS

A subset  $A$  of a semigroup  $S$  is called right unitary if for every  $a \in A$  and for every  $s \in S$ ,  $sa \in A$  implies  $s \in A$ , left unitary if  $a \in A$  and for every  $s \in S$ ,  $as \in A$  implies  $s \in A$  and unitary if it is both left and right unitary. In this section we will define the corresponding concepts in  $\Gamma$ -semigroup amalgams.

**Definition 3.1** A  $\Gamma$ -subsemigroup  $U$  of a  $\Gamma$ -semigroup  $S$  is said to be  $\Gamma$ -unitary in  $S$  if  $\forall u \in U, \gamma \in \Gamma$  and  $s \in S$ ,

$$(i) \quad u\gamma s \in U \text{ implies } s \in U$$

$$(ii) \quad s\gamma u \in U \text{ implies } s \in U$$

Let  $S$  be a  $\Gamma$ -semigroup and  $U$  be a  $\Gamma$ -subsemigroup of  $S$ .

The following sets are empty or  $\Gamma$ -subsemigroup of  $U$ , see [6]:

$$L_\Gamma(S) = \{u \in U : \forall s \in S \setminus U, \gamma \in \Gamma; s\gamma u \in U\}$$

$$L'_\Gamma(S) = \{u \in U : \forall s \in S \setminus U, \gamma \in \Gamma; s\gamma u = s\}$$

$$L''_\Gamma(S) = \{u \in U : \forall s \in S \setminus U, \gamma \in \Gamma; s\gamma u \in S \setminus U\}$$

$$R_\Gamma(S) = \{u \in U : \forall s \in S \setminus U, \gamma \in \Gamma; u\gamma s \in U\}$$

$$R'_\Gamma(S) = \{u \in U : \forall s \in S \setminus U, \gamma \in \Gamma; u\gamma s = s\}$$

$$R''_\Gamma(S) = \{u \in U : \forall s \in S \setminus U, \gamma \in \Gamma; u\gamma s \in S \setminus U\}$$

**Lemma 3.2** Let  $S$  be a  $\Gamma$ -semigroup.  $L_\Gamma(S) = \{u \in U : \forall s \in S \setminus U, \gamma \in \Gamma; s\gamma u \in U\}$  [ $R_\Gamma(S)$ ] is empty or a left[right]  $\Gamma$ -ideal of  $S$ .

**Proof 3.3** Suppose  $L_\Gamma(S)$  is non-empty. For  $u_1, u_2 \in L_\Gamma(S)$ , for all  $s \in S \setminus U$  and  $\gamma \in \Gamma$ , we have  $s\gamma u_1 \in U$  and  $s\gamma u_2 \in U$ .

Now for all  $s \in S \setminus U$  and  $\gamma, \beta \in \Gamma$ ;  $s\gamma(u_1\beta u_2) = (s\gamma u_1)\beta u_2 = u_3\beta u_2 \in U$  where  $u_3 = s\gamma u_1$  implies  $u_1\beta u_2 \in L_\Gamma(S)$ . Hence  $L_\Gamma(S)$  is an  $\Gamma$ -ideal of  $S$ .

Clearly

$$\begin{aligned} L_\Gamma(S) \cap L'_\Gamma(S) &= \{\} \\ L_\Gamma(S) \cap L''_\Gamma(S) &= \{\} \\ R_\Gamma(S) \cap R'_\Gamma(S) &= \{\} \\ R_\Gamma(S) \cap R''_\Gamma(S) &= \{\} \end{aligned}$$

and  $L'_\Gamma(S) \subseteq L''_\Gamma(S)$  and  $R'_\Gamma(S) \subseteq R''_\Gamma(S)$

**Definition 3.4** A  $\Gamma$ -subsemigroup  $U$  of a  $\Gamma$ -semigroup  $S$  is said to be quasi  $\Gamma$ -unitary in  $S$  if  $U = L_\Gamma(S) \cup L''_\Gamma(S) = R_\Gamma(S) \cup R''_\Gamma(S)$ .

**Theorem 3.5** Let  $U$  be a quasi  $\Gamma$ -unitary  $\Gamma$ -subsemigroup of  $S$ .  $U$  is  $\Gamma$ -unitary in  $S$  if and only if  $L_\Gamma(S) = R_\Gamma(S) = \phi$ .

**Proof 3.6** Suppose  $U$  is  $\Gamma$ -unitary in  $S$ , then  $\forall u \in U, \gamma \in \Gamma$  and  $s \in S \setminus U$  we have  $u\gamma s \in U$  implies  $s \in U$  and  $s\gamma u \in U$  implies  $s \in U$ . If possible suppose  $L_\Gamma(S) = \phi$ , then there exist an element  $u \in U$  such that for all  $s \in S \setminus U, \gamma \in \Gamma, s\gamma u \in U$  which is a contradiction to the fact that  $U$  is  $\Gamma$ -unitary. Hence  $L_\Gamma(S) = \phi$ .

Similarly  $R_\Gamma(S) = \phi$ . Conversely suppose  $L_\Gamma(S) = R_\Gamma(S) = \phi$ .

Then for all  $u \in U, s \in S \setminus U, \gamma \in \Gamma, s\gamma u \notin U$ . As a result for all  $s \in S \setminus U$  such that  $s\gamma u \in U$  must imply  $s \in U$ . Similarly since  $R_\Gamma(S) = \phi, s\gamma u \in U$  implies  $s \in U$ . Hence  $U$  is  $\Gamma$ -unitary in  $S$ .

**Theorem 3.7** Let  $U$  be a quasi  $\Gamma$ -unitary  $\Gamma$ -subsemigroup of  $S$ .  $U$  is  $\Gamma$ -ideal in  $S$  if and only if  $L''_\Gamma(S) = R''_\Gamma(S) = \phi$ .

**Proof 3.8** Suppose  $U$  is  $\Gamma$ -ideal in  $S$ , then

$$UTS \subset U, STU \subset U. \quad (1)$$

If possible, assume  $L''_{\Gamma}(S) \neq \phi$ , then there exist a  $u \in U$  such that for all  $s \in S \setminus U, \gamma \in \Gamma, s\gamma u \in S \setminus U$  which is a contradiction to (1). Hence  $L''_{\Gamma}(S) = \phi$ . Similarly  $R''_{\Gamma}(S) = \phi$ .

Conversely suppose  $L''_{\Gamma}(S) = R''_{\Gamma}(S) = \phi$ , then for all  $u \in U, s \in S \setminus U, \gamma \in \Gamma, s\gamma u \notin S \setminus U$ . Hence  $s\gamma u$  must be in  $U$ , i.e,  $S\Gamma U \subset U$ . Similarly  $U\Gamma S \subset U$ . Thus  $U$  is  $\Gamma$ -ideal in  $S$ .

**Lemma 3.9** Let  $U$  be a proper  $\Gamma$ -subsemigroup of a  $\Gamma$ -semigroup  $S$  and let  $s \in S \setminus U, u \in U$ . The following conditions are equivalent:

- (a) if there exist  $s$  such that  $s\gamma u \in U$  then for all  $s, s\gamma u \in U$
- (b) if there exist  $s$  such that  $s\gamma u \in S \setminus U$  then for all  $s, s\gamma u \in S \setminus U$
- (c)  $U = L_{\Gamma}(S) \cup L''_{\Gamma}(S)$

Similarly the dual statements are equivalent:

- (a') if there exist  $s$  such that  $u\gamma s \in U$  then for all  $s, u\gamma s \in U$
- (b') if there exist  $s$  such that  $u\gamma s \in S \setminus U$  then for all  $s, u\gamma s \in S \setminus U$
- (c')  $U = R_{\Gamma}(S) \cup R''_{\Gamma}(S)$

**Proof 3.10** (a)  $\implies$  (b). Let there exist  $t_0(t_0 u \in S \setminus U)$ .

Suppose that there exist  $t_1(t_1 u \in U)$ .

Then by (a),  $t_0 u \in U$  which is a contradiction.

Similarly (b)  $\implies$  (a).

(a) and (b)  $\implies$  (c). If the first part of (a) holds, then  $u \in L_{\Gamma}(S)$ . If the first part of (b) holds, then  $u \in L''_{\Gamma}(S)$ . Since  $U$  is a proper  $\Gamma$ -subsemigroup of  $S$ , either the first part of (a) holds or the first part of (b) holds.

Hence  $u \in L_{\Gamma}(S)$  or  $u \in L''_{\Gamma}(S)$ . That is,  $U = L_{\Gamma}(S) \cup L''_{\Gamma}(S)$ .

(c)  $\implies$  (a). Suppose that there exist  $s$  such that  $s\gamma u \in U$ . Then  $u \notin L''_{\Gamma}(S)$ . By (c),  $u \in L_{\Gamma}(S)$ . i.e, for all  $s, s\gamma u \in U$ . Proceeding as above, conditions (a'), (b') and (c') are equivalent.

Using the above lemma, we see that a proper  $\Gamma$ -subsemigroup  $U$  of a  $\Gamma$ -semigroup  $S$  is quasi  $\Gamma$ -unitary in  $S$  if and only if both (a) and (a') hold, or equivalently, both (b) and (b') hold. This can be used in the embeddability of a  $\Gamma$ -semigroup amalgam.

#### 4. $\Gamma$ -SEMIGROUP AMALGAMS WHOSE PARTIAL $\Gamma$ -GROUPOIDS ARE PRE $\Gamma$ -SEMIGROUPS

Let  $U_1, S_1, S_2$  be mutually disjoint non-empty sets.  $\Gamma_0, \Gamma_1, \Gamma_2$  are non-empty sets such that  $U$  is a  $\Gamma_0$ -semigroup,  $S_1$  is a  $\Gamma_1$ -semigroup,  $S_2$  is  $\Gamma_2$ -semigroup. Then the collection  $\mathbf{A} = [(U, \Gamma_0), (S_1, \Gamma_1), (S_2, \Gamma_2), f_1, f_2]$  constitutes a  $\Gamma$ -semigroup amalgam where  $f_i = (f'_i, f''_i)$  are  $\Gamma$ -monomorphisms such that  $f'_i : U \rightarrow S_i$  and  $f''_i : \Gamma_0 \rightarrow \Gamma_i$  for  $i = \{1, 2\}$  and  $U$  is called core of the amalgam.

A  $\Gamma$ -semigroup amalgam  $\mathbf{A} = [(U, \Gamma_0), (S_1, \Gamma_1), (S_2, \Gamma_2), f_1, f_2]$  is embeddable in another  $\Gamma$ -semigroup  $(T, \Gamma_T)$  if there exist mappings ( $\Gamma$ -monomorphisms)  $g = (g', g'') : (U, \Gamma_0) \rightarrow (T, \Gamma_T)$  and  $g_i = (g'_i, g''_i) : (S_i, \Gamma_i) \rightarrow (T, \Gamma_T)$  where  $g_i : S_i \rightarrow T$  and  $g''_i : \Gamma_i \rightarrow \Gamma_T$  for  $i = 1, 2$  and  $g' : U \rightarrow T$ ,  $g'' : \Gamma_0 \rightarrow \Gamma_T$  such that the following holds :

**[1]** : the diagram

$$\begin{array}{ccc} (U, \Gamma_0) & \xrightarrow{f_1} & (S_1, \Gamma_1) \\ \downarrow f_2 & \searrow f & \downarrow g_1 \\ (S_2, \Gamma_2) & \xrightarrow{f_2} & (T, \Gamma_T) \end{array} \quad \text{commutes and}$$

**[2]** :  $g_1((S_1, \Gamma_1)) \cap g_2((S_2, \Gamma_2)) = g((U, \Gamma_0))$

The second condition can be restated as:

- whenever  $g_1(s_1\gamma_1s'_1) = g_2(s_2\gamma_2s'_2)$  (for  $s_1, s'_1 \in S_1$ ,  $s_2, s'_2 \in S_2$ ,  $\gamma_1 \in \Gamma_1$  and  $\gamma_2 \in \Gamma_2$ ) there exists  $u \in U$  and  $\gamma_0 \in \Gamma_0$  such that  $f'_i(u) = s_i$  and  $f''_i(\gamma_0) = \gamma_i$  for  $i = 1, 2$ .

If the amalgam satisfies condition [1] then it is weakly  $\Gamma$ -embeddable and if the amalgam satisfies both conditions, then it is strongly  $\Gamma$ -embeddable.

Throughout in this paper we consider three  $\Gamma$  semigroups  $U, S_1, S_2$  over the same  $\Gamma$  for an amalgam. A  $\Gamma$ -semigroup amalgam determines a partial  $\Gamma$ -groupoid  $\mathcal{G}_U$  and the amalgam is said to be embeddable in a  $\Gamma$ -semigroup if  $\mathcal{G}_U$  is a partial  $\Gamma$ -semigroup. The partial  $\Gamma$ -groupoid associated with a  $\Gamma$ -semigroup amalgam  $\mathcal{A} = [U, S_1, S_2, f_1, f_2]$  is described by setting:

set  $f_1(U) = U_1$ ,  $f_2(U) = U_2$ ,  $S'_1 = S_1 \setminus U_1 \cup U$  and  $S'_2 = S_2 \setminus U_2 \cup U$ . Define an

operation  $*$  on  $S'_i$  so that the mapping  $\varphi_i : S'_i \rightarrow S_i$  defined by

$$\varphi_i(a) = \begin{cases} a, & \text{if } a \in S_i \setminus U_i. \\ f_i(a), & \text{if } a \in U. \end{cases}$$

is a  $\Gamma$ -monomorphism of  $S'_i$  onto  $S_i$ . Assume  $S'_1 \cap S'_2 = U$  and the operation  $*$  defined on  $S'_i$  coincides with the operation of the  $\Gamma$ -semigroup  $U$  when restricted to  $U$ . Thus  $(U, *)$  is a  $\Gamma$ -subsemigroup of  $(S'_i, *)$ .  $M = S'_1 \cup S'_2$  and  $*$  is the operation on  $M$  whenever it is defined, then  $(M, \Gamma, D)$  is a partial  $\Gamma$ -groupoid which we denote by  $\mathcal{G}_U$ .

**Lemma 4.1** *For any  $\Gamma$ -semigroup amalgam  $\mathcal{A} = [U; S_1; S_2, f_1; f_2]$  the partial  $\Gamma$ -groupoid  $\mathcal{G}_U$  satisfies axioms (M3), (M4) and (M5).*

**Proof 4.2** *Suppose  $(w, \alpha, x, \beta, y, \gamma, z)_D$ ,  $w, x, y, z \in M$ ,  $\alpha, \beta, \gamma \in \Gamma$  and  $x, y \in S'_i$  for some  $i = 1, 2$ . If  $x\beta y \in U$  then  $(w, \alpha, x\beta y)_D$  and  $(x\beta y, \gamma, z)_D$ . Thus (M3) holds. Now suppose  $x\beta y \in S'_i \setminus U$  then either  $x \in S'_i \setminus U$  or  $y \in S'_i \setminus U$ . If  $x \in S'_i \setminus U$ , we have  $x \in S'_i$  since  $(w, \alpha, x)_D$ . Therefore  $(w, \alpha, x\beta y)_D$ . Similarly, if  $y \in S'_i \setminus U$ , then  $(x\beta y, \gamma, z)_D$ . Thus (M3) holds.*

*To prove that (M4) holds, suppose  $w, x, y, z \in M$ ,  $\alpha, \beta, \gamma \in \Gamma$ ,  $(w, \alpha, y, \beta, z)_D$ ,  $(w, \gamma, x\alpha y)_D$  and  $x, y \in S'_i$ . If  $x\alpha y \in U$  then  $(x\alpha y, z)_D$ . Thus (M4) holds. Now suppose  $x\alpha y \in S'_i \setminus U$ . Since  $(w, \gamma, x\alpha y)_D$ , we have  $w \in S'_i$ . Therefore  $(w, \gamma, x)_D$  and hence (M4) holds.*

*Similarly, (M5) holds.*

The following theorem provides a detail explanation about  $\Gamma$ -semigroup amalgams that are embeddable and have the property that, in the free products amalgamating the common  $\Gamma$ -subsemigroups, equivalent reduced words have the same length.

**Theorem 4.3** *Let  $\mathcal{A} = [U, S_1, S_2, f_1, f_2]$  be a  $\Gamma$ -semigroup amalgam. Then the partial  $\Gamma$ -groupoid  $\mathcal{G}_U$  is a pre- $\Gamma$ -semigroup if and only if it satisfies the following conditions:*

- (i)  $U$  is a quasi  $\Gamma$ -unitary  $\Gamma$ -subsemigroup of  $S'_1$  and  $S'_2$ .
- (ii)  $L_{\Gamma_i} = R_{\Gamma_j}$  for all  $i, j \in \{1, 2\}$  with  $i \neq j$ .
- (iii)  $s_i(us_j) = (s_iu)s_j$  for all  $s_i \in S'_i \setminus U$ ,  $s_j \in S'_j \setminus U$ ,  $u \in L_{\Gamma_i}$  and all  $i, j \in \{1, 2\}$  with  $i \neq j$ .

This theorem can be restated in a more general way as follows :

**Theorem 4.4** Let  $\mathcal{A} = [U, S_1, S_2, f_1, f_2]$  be a  $\Gamma$ -semigroup amalgam. If  $U$  is a  $\Gamma$ -unitary  $\Gamma$ -subsemigroup of  $S'_1$  and  $S'_2$  then  $\mathcal{G}_U$  is a pre- $\Gamma$ -semigroup.

**Proof 4.5** By Theorem 2.6 the partial  $\Gamma$ -groupoid  $\mathcal{G}_U$  is a pre- $\Gamma$ -semigroup if and only if it satisfies axioms (M1) to (M5). By Lemma 4.1,  $\mathcal{G}_U$  satisfies (M3), (M4) and (M5). Suppose  $x, y, z \in M$ ,  $(x, \alpha, y, \beta, z)_D$  and  $x, y \in S'_i$  for some  $i = 1, 2$ . If  $z \in S'_i$  then  $x, y, z$  belongs to the same  $\Gamma$ -semigroup, so (M1), (M2) holds. Now suppose  $z \notin S'_i$ , i.e.,  $z \in S'_j$  and  $i \neq j$ , since  $(y, \beta, z)_D$  we have  $y \in U$ . If  $x \in U$  then  $x, y, z$  belongs to the same  $\Gamma$ -semigroup, implies (M1), (M2) holds. If  $x \in S'_i \setminus U$ , then  $x\alpha y \in S'_i \setminus U$  since  $U$  is  $\Gamma$ -unitary in  $S'_i$  and  $y\beta z \in S'_j \setminus U$  since  $U$  is  $\Gamma$ -unitary in  $S'_j$ . Consequently  $(x\alpha y, \beta, z)_D$  and  $(x, \alpha, y\beta z) \notin D$  and so (M1) and (M2) hold.

As a corollary to the above Theorem 4.4, we have

**Corollary 4.6** Let  $\mathcal{A} = [U, S_1, S_2, f_1, f_2]$  be a  $\Gamma$ -semigroup amalgam. If  $U$  is a  $\Gamma$ -unitary  $\Gamma$ -subsemigroup of  $S'_1$  and  $S'_2$  then  $\mathcal{G}_U$  is a partial  $\Gamma$ -semigroup.

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