

A New Simple Example of an Atomic Domain Which is not ACCP

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Abstract

We give a new simple example of an atomic domain which is not ACCP. Our example is a monoid domain $F[X; M]$, where F is a field and M is a submonoid of the additive monoid of nonnegative rational numbers.

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1. INTRODUCTION

It is well-known that in an ACCP domain (i.e., a domain in which every increasing sequence of principal ideals is stationary) every non-zero non-unit element can be written as a finite product of irreducible elements (also called atoms). “Somewhat surprisingly”, to quote [1], “the converse is not true; an atomic domain need not satisfy ACCP, but examples are hard to come by.” The first such example is due to A. Grams in [6], where she also mentioned that P. M. Cohn conjectured that another domain is atomic but not ACCP. (Note that in [3] Cohn assumed that these two notions are equivalent.) A new type of examples was given by A. Zaks in [9], where he also proved Cohn’s conjecture. The goal of this note is to give a new simple example of an atomic non-ACCP domain. Our example is a monoid domain $F[X; M]$, where F is a field and M is a submonoid of the additive monoid of nonnegative rational numbers. We would like to mention that, in general, it is much easier to work with the localization of $F[X; M]$ with respect to its maximal ideal consisting of “polynomials” with zero constant term, than with $F[X; M]$ itself, nevertheless our example is fairly simple.

2. NOTATION AND PRELIMINARIES

We begin by recalling some definitions and statements. All the notions that we use but not define in this paper can be found in the classical reference books [2] by P. M. Cohn, [4] by R. Gilmer, [7] by I. Kaplansky, and [8] by D. G. Northcott.

We use \subseteq to denote inclusion, \subset to denote strict inclusion, and we denote $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $\mathbb{Q}_+ = \{q \in \mathbb{Q} \mid q \geq 0\}$.

An *integral domain* is a commutative ring $R \neq \{0\}$ (with multiplicative identity 1) such that for any $x, y \in R$, $xy = 0$ implies $x = 0$ or $y = 0$. All the rings that we use in this paper are assumed to be commutative and with multiplicative identity 1. In fact, all of them will be integral domains. An element $a \in R$ is said to be a *unit* of R if it has a multiplicative inverse in R . A non-zero non-unit element $a \in R$ is said to be *irreducible* (and called an *atom*) of R if $a = bc$ ($b, c \in R$) implies that at least one of the elements b, c is a unit. Two elements $a, b \in R$ are said to be *associates* if $a = ub$, where u is a unit. We then write $a \sim b$. An integral domain R is said to be *atomic* if every non-zero non-unit element of R can be written as a finite product of atoms. An integral domain R is said to be an *ACCP domain* if every increasing sequence

$$(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \dots$$

of principal ideals of R is stationary, meaning that $(a_n) = (a_{n+1}) = (a_{n+2}) = \dots$ for some n .

Many notions related to factorization in an integral domain can be already defined in the underlying multiplicative monoid (R, \cdot) . Hence we can generalize them by defining them in an arbitrary commutative monoid $(M, +)$ by just translating the terminology from the multiplicative one to the additive one. The next definitions illustrate this point of view.

A *commutative monoid*, written additively, is a nonempty set M with an operation $+$: $M \times M \rightarrow M$ which is associative, commutative, and has an identity element called *zero* (i.e., an element $0 \in M$ such that $a + 0 = a$ for every $a \in M$). All the monoids used in this paper are assumed to be commutative and written additively. From now on we call them just *monoids*. A subset M' of a monoid M is called a *submonoid* of M if $0 \in M'$ and if for every $a, b \in M'$ we have $a + b \in M'$. If A is a subset of a monoid M , then $\langle A \rangle$ denotes the submonoid of M *generated* by A , i.e., the smallest submonoid of M which contains A . A subset I of a monoid M is called an *ideal* of M if $M + I = I$, i.e., if for every $a \in I$, $M + a \subseteq I$. (Here $S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$ for any

two subsets S_1, S_2 of M .) Note that \emptyset is an ideal of M (the smallest ideal of M) and that the ideal of M generated by \emptyset is \emptyset . An ideal I of M is said to be *principal* if there is an element $a \in I$ such that $I = M + a$. We then write $I = (a)$. An element $a \in M$ is said to be a *unit* of M if it has an (additive) inverse in M . Two elements $a, b \in M$ are said to be *associates* if $a = u + b$, where u is a unit. We then write $a \sim b$. A non-unit element $a \in M$ is said to be *irreducible* (and called an *atom* of M) if $a = b + c$ ($b, c \in M$) implies that at least one of the elements b, c is a unit. A monoid M is said to be *atomic* if every non-unit element of M can be written as a finite sum of atoms. A monoid M is said to be an *ACCP monoid* if every increasing sequence

$$(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \dots$$

of principal ideals of M is stationary.

As we said in Introduction, it is well-known that *an ACCP domain is atomic*. This is in fact a property of the underlying multiplicative monoid of the domain, so *an analogous statement holds for monoids as well*. The converse is not true, as we are going to see in our theorem in the next section.

If M is a monoid and F is a field, we will consider the *monoid ring* $F[X; M]$ associated with M . It consists of the polynomial expressions (also called polynomials)

$$f = \alpha_1 X^{a_1} + \alpha_2 X^{a_2} + \dots + \alpha_n X^{a_n}, \quad (1)$$

where $n \geq 1$, $\alpha_i \in F$, and $a_i \in M$ ($i = 1, 2, \dots, n$). In all the situations in which we will be considering $F[X; M]$, M is a submonoid of the monoid \mathbb{Q}_+ . Then the monoid ring $F[X; M]$ is an integral domain (we will say a *monoid domain*), as it is easy to see, and the units of $F[X; M]$ are precisely the elements of F . Also we will assume that f is always written with decreasing exponents, i.e., $a_1 > a_2 > \dots > a_n$, so that we can define the *degree* of f , $\deg(f)$, to be a_1 (assuming that $\alpha_1 \neq 0$). By convention $\deg(0) = -\infty$.

3. THE EXAMPLE

Proposition 3.1. *Let M be a submonoid of $(\mathbb{Q}_+, +)$. Then the monoid domain $F[X; M]$ is ACCP if and only if the monoid M is ACCP.*

Proof. Suppose $F[X; M]$ is ACCP. Consider an increasing sequence

$$(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \dots$$

of principal ideals of M . It corresponds to it an increasing sequence

$$(X^{a_1}) \subseteq (X^{a_2}) \subseteq (X^{a_3}) \subseteq \dots$$

of principal ideals of $F[X; M]$. This sequence is stationary since $F[X; M]$ is ACCP, hence there is an $n \in \mathbb{N}$ such that

$$(X^{a_n}) = (X^{a_{n+1}}) = (X^{a_{n+2}}) = \dots$$

Hence

$$X^{a_n} \sim X^{a_{n+1}} \sim X^{a_{n+2}} \sim \dots$$

Since the units of $F[X; M]$ are precisely the elements of F , we have

$$a_n = a_{n+1} = a_{n+2} = \dots,$$

hence

$$(a_n) = (a_{n+1}) = (a_{n+2}) = \dots$$

Hence M is ACCP.

Conversely, suppose that M is ACCP. Consider an increasing sequence

$$(f_1) \subseteq (f_2) \subseteq (f_3) \subseteq \dots \quad (2)$$

of principal ideals of $F[X; M]$. For each $i = 1, 2, 3, \dots$ let

$$f_i = \alpha_{i,1}X^{a_{i,1}} + \alpha_{i,2}X^{a_{i,2}} + \dots + \alpha_{i,n_i}X^{a_{i,n_i}}$$

with

$$a_{i,1} > a_{i,2} > \dots > a_{i,n_i}.$$

For each $i = 1, 2, 3, \dots$ there is a $g_i \in F[X; M]$ such that

$$f_i = f_{i+1}g_i. \quad (3)$$

If $\deg(g_i) = b_{i,1}$ for $i = 1, 2, 3, \dots$, then

$$a_{i,1} = a_{i+1,1} + b_{i,1}, \quad (4)$$

hence we have an increasing sequence

$$(a_{1,1}) \subseteq (a_{2,1}) \subseteq (a_{3,1}) \subseteq \dots$$

of principal ideals of M . Since M is ACCP, this sequence is stationary, hence for some n we have

$$(a_{n,1}) = (a_{n+1,1}) = (a_{n+2,1}) = \dots$$

This in turn implies that for every $j \geq 0$,

$$\begin{aligned} a_{n+j,1} &= a_{n+j+1,1} + c_{n+j}, \\ a_{n+j+1,1} &= a_{n+j,1} + c_{n+j+1}, \end{aligned}$$

for some $c_{n+j}, c_{n+j+1} \in \mathbb{Q}_+$. Hence $c_{n+j} = 0$ for all $j \geq 0$, so that

$$a_{n,1} = a_{n+1,1} = a_{n+2,1} = \dots$$

It now follows from (4) that

$$0 = b_{n,1} = b_{n+1,1} = b_{n+2,1} = \dots,$$

hence, due to (3), $f_i \sim f_{i+1}$ for every $i \geq n$. Hence the sequence (2) is stationary and so $F[X; M]$ is ACCP. \square

The next theorem is our example of a monoid domain which is not an ACCP domain.

Theorem 3.2. *Let*

$$p_1 = 2, p_2 = 3, p_3 = 5, \dots$$

be the sequence of prime numbers. Consider the submonoid

$$M = \left\langle \frac{1}{p_1 p_3}, \frac{1}{p_2 p_4}, \frac{1}{p_3 p_5}, \dots \right\rangle$$

of $(\mathbb{Q}_+, +)$. Then the monoid domain $F[X; M]$ is atomic, but not ACCP.

Proof. To show that M is atomic, it is enough to show that each $\frac{1}{p_i p_{i+2}}$ ($i \geq 1$) is an atom of M . Suppose to the contrary, i.e., that there is an $i \in \mathbb{N}$ such that $\frac{1}{p_i p_{i+2}}$ is not an atom. Then

$$\frac{1}{p_i p_{i+2}} = a + b,$$

where a, b are two nonzero elements of M , both smaller than $\frac{1}{p_i p_{i+2}}$. Hence

$$a = \frac{k_1}{p_{\alpha_1} p_{\alpha_1+2}} + \dots + \frac{k_r}{p_{\alpha_r} p_{\alpha_r+2}}, \quad (5)$$

$$b = \frac{l_1}{p_{\beta_1} p_{\beta_1+2}} + \dots + \frac{l_s}{p_{\beta_s} p_{\beta_s+2}}, \quad (6)$$

where $k_1, \dots, k_r \in \mathbb{N}$, $l_1, \dots, l_s \in \mathbb{N}$, $i < \alpha_1 < \dots < \alpha_r$, and $i < \beta_1 < \dots < \beta_s$. However when we add all the fractions from (5) and (6), using the product of the

denominators that appear in (5) and (6) as a common denominator, we don't get p_i in the denominator of the sum. That's a contradiction, so each $\frac{1}{p_i p_{i+2}}$ is an atom.

Now consider the following sequence of principal ideals of M :

$$\left(\frac{1}{2} + \frac{1}{3}\right), \left(\frac{1}{3} + \frac{1}{5}\right), \left(\frac{1}{5} + \frac{1}{7}\right), \dots, \left(\frac{1}{p_i} + \frac{1}{p_{i+1}}\right), \dots$$

We have

$$\left(\frac{1}{p_i} + \frac{1}{p_{i+1}}\right) \subseteq \left(\frac{1}{p_{i+1}} + \frac{1}{p_{i+2}}\right)$$

for $i = 1, 2, 3, \dots$ since

$$\frac{1}{p_i} + \frac{1}{p_{i+1}} = \frac{1}{p_{i+1}} + \frac{1}{p_{i+2}} + \frac{p_{i+2} - p_i}{p_i p_{i+2}}.$$

This sequence is not stationary, so M is not ACCP. Hence, by Proposition 3.1, $F[X; M]$ is not ACCP.

We now show that $F[X; M]$ is atomic. Suppose to the contrary. Then there is an element $f \in F[X; M]$ such that

$$\begin{aligned} f &= f_0 f_1 \\ &= f_0 f_{10} f_{11} \\ &= f_0 f_{10} f_{110} f_{111} \\ &= \dots, \end{aligned}$$

where $f_0, f_1, f_{10}, f_{11}, f_{110}, f_{111}, \dots$ are non-constant polynomials. Let β be the largest exponent of f and $\beta_{111\dots10}$ (resp. $\beta_{111\dots11}$) the largest exponents of the polynomials $f_{111\dots10}$ (resp. $f_{111\dots11}$). Since

$$\begin{aligned} \beta &= \beta_0 + \beta_1 \\ &= \beta_0 + \beta_{10} + \beta_{11} \\ &= \beta_0 + \beta_{10} + \beta_{110} + \beta_{111} \\ &= \dots, \end{aligned}$$

no atom of M can appear as an addend in infinitely many $\beta_{111\dots10}$. Let p_i be the largest prime that appears in the denominator of β , written in reduced form. There is an n big enough so that when we write $\beta_0 + \beta_{10} + \beta_{110} + \dots + \beta_{111\dots10}$, with n ones in the last index, as a sum of atoms, the atom $\frac{1}{p_{j-2} p_j}$ with the largest prime p_j has $j > i$ and appears in the sum less than p_j times (otherwise the sums $\beta_0 + \beta_{10} + \beta_{110} + \dots + \beta_{111\dots10}$ would become arbitrarily large). Then this sum of atoms, when written as a fraction in reduced form, would have p_j in the denominator and so it could not be equal to β , a contradiction. Thus $F[X; M]$ is atomic. \square

4. CONCLUDING REMARKS

Let M be a submonoid of $(\mathbb{Q}_+, +)$ and F a field. If the monoid domain $F[X; M]$ is atomic, then the monoid M is atomic, as it was proved in [5, Proposition 2.10]. We would like to mention Question 2.11 from [5] which asks if the opposite direction is also true. In other words, is it true that $F[X; M]$ is atomic if and only if M is atomic. (The analogous statement for the ACCP property is true, as we proved in the above Proposition 3.1.) In our example in Theorem 3.2 both the monoid M and the monoid domain $F[X; M]$ are atomic.

More generally, we can ask for which cancellative torsion-free commutative monoids $(M, +)$ the condition M atomic is equivalent to the condition $F[X; M]$ atomic (F a field).

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