Overview of the Construction of Irreducible Representations of Symmetric Groups

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Abstract

Representation theory of symmetric groups is one of the subjects that has been studied for centuries. As such, there exists lots of expositions, surveys and research articles whose objective is to illuminate further on this subject. The goal of this article is to introduce this topic to a young graduate student by providing a self contained exposition of the topic in a manner that is not too elementary nor too advanced. We hope that readers will use this article as a preparation towards reading much more advanced and voluminous research materials.

AMS subject classification:

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1. Introduction

It is well known that the number of irreducible representations is equal to the number of the distinct conjugacy classes of $S_n$ (see for example [9]). On the other hand, if $g \in S_n$ is a representative of a conjugacy class, then its cycle-type corresponds to a partition $\lambda$ of $n$ ([8, Prop. 2.32]). It follows therefore that the number of distinct conjugacy classes of $S_n$ is equal to the number of unordered partitions of $n$ and further that each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$ corresponds to an irreducible $\mathbb{C}[S_n]$-module.

Our main goal is to use an example to illustrate how to construct an irreducible $\mathbb{C}[S_n]$-module. We will use the following procedure: for each partition $\lambda \vdash n$, we construct as subgroup $S_\lambda \subset S_n$, next we induce the trivial representation of $S_\lambda$ up to $S_n$ and denote by $\rho_\lambda$ the corresponding representation of $S_n$. Consequently, we get a $\mathbb{C}[S_n]$-module $M_\lambda$ corresponding to the representation $\rho_\lambda$.

The $\mathbb{C}[S_n]$-module $M_\lambda$ is not irreducible in general, however, we can defined an order on the set $\{\lambda^{(1)}, \lambda^{(2)}, \ldots\}$ of unordered partitions $n$ in such a way that the first module
$M^{(1)}_\lambda$ is irreducible and we denote it by $S^{(1)}_\lambda$, the second module $M^{(2)}_\lambda$ decomposes into copies of $S^{(1)}_\lambda$ and a new irreducible module $S^{(2)}_\lambda$, and in general, each module $M^{(j)}_\lambda$ will decompose into a number of copies of $S^{(i)}_\lambda$ for $i < j$ and a new irreducible module $S^{(j)}_\lambda$. In the end we get a sequence of irreducible $\mathbb{C}[S_n]$ modules $S^{(1)}_\lambda, S^{(2)}_\lambda, S^{(3)}_\lambda, \ldots$ called the Specht Modules.

2. Young Diagrams and Tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of a non-negative integer $n$. We will assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. The Young diagram of $\lambda$ is obtained by stacking sequences of boxes in such a way that the top-most row has $\lambda_1$ square boxes, the second top row has $\lambda_2$ square boxes, and in general, the $i$-th row from the top will have $i$ boxes. For example, the Young diagram of the partition $\lambda = (3, 2, 1) \vdash 7$ is the diagram

```
\[ \begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array} \]
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A Young tableau of shape $\lambda$ (also called $\lambda$-tableau) is a labeled Young diagram, where the boxes are labeled with the numbers 1 to $n$ in a bijective manner. For example if $n = 3$ and $\lambda = (2, 1)$, then some of the possible Young tableaux of shape $\lambda$ are

\[
\begin{align*}
t_1 &= \begin{bmatrix} 1 & 2 & 3 \\ 3 & \end{bmatrix}, & t_2 &= \begin{bmatrix} 1 & 3 & 2 \\ 2 & \end{bmatrix}, & t_3 &= \begin{bmatrix} 2 & 3 & 1 \\ 1 & \end{bmatrix}.
\end{align*}
\]

Let $T_\lambda$ be the set of all Young tableaux of shape $\lambda$. A tableau $t_1 \in T_\lambda$ is said to be row equivalent to another tableau $t_2 \in T_\lambda$ if for each $i$ the set of elements in row $i$ of $t_1$ is equal to the set of elements in row $i$ of $t_2$.

**Definition 2.1.** Let $\lambda$ be a partition of $n$, $T_\lambda$ be the set of all Young tableaux of shape $\lambda$ and denote by $\cong$ the row equivalence of tableaux in $T_\lambda$. A tabloid of shape $\lambda$ is an element in the set $T_\lambda/\cong$. We denote by $[t]$ the equivalence class (tabloid) of a tableau $t \in T_\lambda$.

**Definition 2.2.** Let $[t_1], [t_2], \ldots, [t_r]$ be the complete list of elements in $T_\lambda/\cong$. The permutation module corresponding to $\lambda$ is defined to be

\[
M^\lambda = \mathbb{C}\{[t_1], [t_2], \ldots, [t_r]\}.
\]

An element of $M^\lambda$ is called a polytabloid.

**Example 2.3.** Let $\lambda = (2, 1) \vdash 3$. Then $M^\lambda$ is the complex vector space of rank 3 spanned by the tabloids $[t_1], [t_2]$ and $[t_3]$ where

\[
\begin{align*}
t_1 &= \begin{bmatrix} 2 & 3 & 1 \\ & & \\
\end{bmatrix}, & t_2 &= \begin{bmatrix} 1 & 3 & 2 \\ & & \\
\end{bmatrix}, & t_3 &= \begin{bmatrix} 1 & 2 & 3 \\ & & \\
\end{bmatrix}.
\end{align*}
\]
3. Constructing a Specht Module Corresponding to a Partition

There is a natural action of $T_{\lambda}$ by $S_n$ which can be defined as follows. For any $\sigma \in S_n$ and any $t \in T_{\lambda}$, let $\sigma t$ be the element of $T_{\lambda}$ whose entry at row $i$ and column $j$ is the number obtained by letting $\sigma$ act on the number in row $i$ and column $j$ of $t$. For example, if $\sigma = (132) \in S_3$ then

$$\sigma \begin{pmatrix} 1 & 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 \end{pmatrix}. \quad (4)$$

Given a tableau $t$ of shape $\lambda$ let $C_j$ be the set whose elements are the entries in column $j$ of $t$. Denote by $S_{C_j}$ the permutation group of elements of $C_j$. The column stabilizer $C(t)$ of a $\lambda$-tableau $t$ is defined to be the subgroup of $S_n$ that fixes the columns of $t$ setwise. If $t$ has columns $C_1, C_2, \ldots, C_l$ then its column stabilizer is the subgroup

$$C(t) = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_l}.$$

Example 3.1. Let $\lambda = (2, 2, 1) \vdash 5$ and consider the following tableau of shape $\lambda$.

$$t = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 \end{pmatrix}. \quad (5)$$

Then the column stabilizer subgroup of $t$ is

$$C(t) = S_{\{1, 3, 5\}} \times S_{\{2, 4\}}$$
$$= \{t, (13), (15), (35), (135), (153)\} \times \{t, (24)\}$$
$$= \{t, (13), (15), (35), (135), (24), (13)(24), (15)(24)\}$$
$$\quad (35)(24), (135)(24), (153)(24)\}.$$

It is a trivial exercise to check that $C(t)$ is indeed a subgroup of $S_5$ and additionally, that the action of any element of $C(t)$ on $t$ fixes the columns of $t$ setwise.

Definition 3.2. Let $t$ be a $\lambda$-tableau. The signed column sum of $t$ is the element of $\mathbb{C}[S_n]$ defined by

$$\kappa_t = \sum_{\sigma \in C(t)} \text{sgn}(\sigma) \sigma \quad (6)$$

where $\text{sgn}(\sigma)$ is the signature of the permutation $\sigma$.

At this point, we have all the ingredients required to construct the irreducible representations. For each partition $\lambda$ we already know how to obtain the permutation module $M^\lambda$. The module $M^\lambda$ is not irreducible in general however it has an irreducible submodule $S^\lambda$. The following procedure is used to obtain an irreducible submodule $S^\lambda$ of $M^\lambda$.

Procedure: For each $\lambda \vdash n$ and let $[\tau]$ be a tabloid of shape $\lambda$. For each $t \in [\tau]$
• construct the column stabilizer $C(t)$ for $t$ and let $\kappa_t$ be the signed column sum for $t$.
• set $v_t = \kappa_t t$.

The irreducible representation of $S_n$ corresponding to $\lambda$ is the submodule of $M^\lambda$ generated by the basis $\{v_t : t \in [\tau]\}$.

In order to obtain the characterization of the submodules $S^\lambda$ as described in the introductory section of this article, we shall have to order the set of all unordered partitions of $n$ in it the reversed lexicographical order (see [6, Def. 3.4]). We shall illustrate this using a concrete example in the next section.

4. Concrete Example

We will use a small concrete example ($n = 3$) to illustrate the construction illuminated above. The $\mathbb{C}[S_3]$ module has the following basis elements

$$\{\iota, (12), (13), (23), (123), (132)\}$$

where $\iota$ is the identity permutation. The unordered partitions of 3 are $\lambda = (3), \mu = (2, 1)$ and $\xi = (1, 1, 1)$ listed in the reversed lexicographical order. The corresponding Young diagrams are as follows

$$\lambda \mapsto \begin{array}{c}
\text{[ ]}
\end{array}, \quad \mu \mapsto \begin{array}{c}
\text{[ [] ]}
\end{array}, \quad \xi \mapsto \begin{array}{c}
\text{[ [ ] ]}
\end{array}.$$  \hfill (8)

1. There are six Young tableau of shape $\lambda$ all of which are row equivalent to the $\lambda$ tableau

$$1 | 2 | 3.$$  \hfill (9)

Thus $M^\lambda$ is one dimensional and is therefore an irreducible submodule of $\mathbb{C}[S_3]$. Thus in this case we have $S^\lambda = M^\lambda$.

2. There are three $\mu$-tabloids. These are

$$[\tau] = \left\{ \begin{array}{c}
\text{[ ] [ ]}
\end{array} \right\}, \quad [\xi] = \left\{ \begin{array}{c}
\text{[ [ ] ]}
\end{array} \right\}, \quad [\eta] = \left\{ \begin{array}{c}
\text{[ [ ] ]}
\end{array} \right\}.$$  \hfill (10)

Thus $M^\mu$ is a submodule of $\mathbb{C}[S_3]$ of rank 3 generated by $\tau_1 \simeq (12), \tau_2 \simeq (13)$ and $\tau_3 \simeq (23)$. Applying the procedure described in the previous section to the first tabloid we have

By identifying $v_{\tau_1}$ with the element $(12) - (32) \in \mathbb{C}[S_3]$ and $v_{\tau_1}$ with $(21) - (31) \in \mathbb{C}[S_3]$ then $v_{\tau_1}, v_{\tau_1}$ generate a rank 2 submodule of $M^\mu$. The claim is the the generated submodule is irreducible and we denote it by $S^\mu$. Repeating the same procedure on $[\xi]$ and $[\eta]$ then one obtains two dimensional irreducible submodule canonically isomorphic to $S^\mu$. 
3. Lastly, there are six inequivalent $\xi$-tableaux. These are

$$\xi_1 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}, \quad \xi_2 = \begin{array}{c} 1 \\ 3 \\ 2 \end{array}, \quad \xi_3 = \begin{array}{c} 2 \\ 1 \\ 3 \end{array}, \quad \xi_4 = \begin{array}{c} 2 \\ 3 \\ 1 \end{array}, \quad \xi_5 = \begin{array}{c} 3 \\ 1 \\ 2 \end{array}, \quad \xi_6 = \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \quad \text{(11)}$$

Thus $M^\xi$ is of rank 6 and therefore equal to $\mathbb{C}[S_3]$. Further, the column stabilizer for any of the above $\xi_i$ is the entire group $S_3$ and consequently the signed column sum is

$$\iota - (12) - (13) - (23) + (123) + (132)$$

Applying this column sum to the first $\xi$-tableau one obtains the element of $\mathbb{C}[S_3]$ given by

$$v_{\xi_1} = \begin{array}{c} 2 \\ 1 \\ 3 \end{array} - \begin{array}{c} 3 \\ 1 \\ 2 \end{array} - \begin{array}{c} 2 \\ 3 \\ 1 \end{array} + \begin{array}{c} 1 \\ 2 \\ 3 \end{array} + \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

The element $v_{\xi_1}$ spans a rank 1 submodule of $\mathbb{C}[S_3]$ and is therefore irreducible.

5. Conclusion

Representations of symmetric group is an interesting subject to study. This short exposition is meant to introduce a young graduate student to this interesting subject. Even more interesting is the study of modular representations of symmetric group. It can be shown easily that any irreducible representation of symmetric group is equivalent to a matrix representation whose entries are all integral. An interesting question to ask is the following: given a prime number $p$, which of the matrix representations remain irreducible when the entries are reduced modulo $p$?

This is a question which was pioneered by Carter and Lusztig in their paper [1]. In particular, they conjectured a necessary and sufficient condition needed to be satisfied by $\lambda$ and $p$ for the corresponding matrix representation to remain irreducible. In [4], James prove the necessary part of Carter’s conjecture and later [5] together with Mathas classified the irreducible Specht modules for the case $p = 2$. Further, they conjecture a necessary and sufficient condition imposed on the Young diagram of a partition $\lambda$ in order for the corresponding Specht module to be reducible. In [7], Lyle proved a major part of the necessary condition in the conjecture by James and Mathas. The result of Lyle was later used by Fayers [2, 3] to complete the proof of James-Mathas conjecture.
References


