On soft $\beta-$open sets

V.E. Sasikala\textsuperscript{1} and D. Sivaraj

Meenakshi Academy of Higher Education and Research,
Meenakshi University, Chennai,
Tamil Nadu, India.
E-mail: sasikala.rupesh@gmail.com

Abstract

Soft $\beta$-open sets and soft $\beta$-closed sets in a soft topological space are studied. Properties and characterizations of soft $\beta$-closed sets and soft $\beta$-open sets are derived. Its relation with other sets in the soft topological space are also studied.

AMS subject classification: 54A10, 54A05.
Keywords: soft set, soft topology, soft open sets, soft closed sets, soft interior, soft closure.

1. Introduction

To solve several practical problems in economics, engineering, social science and medical science, Molodtsov [4] introduced the theory of soft sets as a new mathematical application for handling few vagueness which the traditional mathematical tool could not resolve. Shabir and Naz [15] studied the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. Also, the authors introduced various soft separations axioms and studied their properties.

Recently, several research papers were published on soft set theory and their applications [14]. Maji et al. [10] elaborated on the operations on soft sets, and some basic properties of these operations have also been revealed. Several of these research papers namely [1–3, 5–13, 16, 17] forms the basis for the theoretical aspects to apply topology on soft sets and help the development of information system and engineering. Few other authors defined over an initial universe set with a fixed set of parameters. Our treatment here is different from that of in paper[15].

\textsuperscript{1}Corresponding author.
2. Preliminaries

The following definitions are essential for the development of the paper.

**Definition 2.1.** [4] Let $U$ be an initial universe and $E$ be a set of parameters. Let $P(U)$ denote the power set of $U$. The pair $(F, E)$ or simply $F_E$, is called a soft set over $U$, where $F$ is a mapping given by $F : E \rightarrow P(U)$. In other words, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $e \in U$, $F(e)$ may be considered as the set of $e$-approximate elements of the soft set $F$. The collection of all soft sets over $U$ and $E$ is denoted by $S(U)$. If $A \subset E$, then the pair $(F, A)$ or simply $F_A$, is called a soft set over $U$, where $F$ is a mapping $F : A \rightarrow P(U)$. Note that for $e \notin A$, $F(e) = \emptyset$.

**Definition 2.2.** [10] The union of two soft sets of $F_B$ and $G_C$ over the common universe $U$ is the soft set $H_D$, where $D = B \cup C$ and for all $e \in D$, $H(e) = F(e)$ if $e \in B - C$, $H(e) = G(e)$ if $e \in C - B$ and $H(e) = F(e) \cup G(e)$ if $e \in B \cap C$, we write $F_B \tilde{\cup} G_C = H_D$.

**Definition 2.3.** [10] The intersection of two soft sets of $F_B$ and $G_C$ over the common universe $U$ is the soft set $H_D$, where $D = B \cap C$ and for all $e \in D$, $H(e) = F(e) \cap G(e)$ if $D = B \cap C$. We write $F_B \tilde{\cap} G_C = H_D$.

**Definition 2.4.** [10] Let $F_B$ and $G_C$ be soft sets over a common universe set $U$ and $B, C \subset E$. Then $F_B$ is a soft subset of $G_C$, denoted by $F_B \tilde{\subset} G_C$, if (i) $B \subset C$ and (ii) for all $e \in B$, $F(e) = G(e)$. Also, $G_C$ is called the soft super set of $F_B$ and is denoted by $F_B \tilde{\supset} G_C$.

**Definition 2.5.** [10] The soft sets $F_B$ and $G_C$ over a common universe set $U$ are said to be soft equal, if $F_B \tilde{\subset} G_C$, and $F_B \tilde{\supset} G_C$. Then we write $F_B = G_C$.

**Definition 2.6.** [10] A soft set $F_B$ over $U$ is called a null soft set denoted by $F_\emptyset$, if for all $e \in B$, $F(e) = \emptyset$.

**Definition 2.7.** [9] The relative complement of a soft set $F_A$, denoted by $F_A^c$, is defined by the approximate function $f_A^c(e) = f_A^c(e)$, where $f_A^c(e)$ is the soft complement of the soft set $f_A(e)$, that is $f_A^c(e) = U - f_A(e)$ for all $e \in E$. It is easy to see that $(F_A^c)^c = F_A$, $F_\emptyset^c = F_E$ and $F_E^c = F_\emptyset$.

**Definition 2.8.** [10] Let $U$ be an initial universe and $E$ be a set of parameters. If $B \subset E$, the soft set $F_B$ over $U$ is called an absolute soft set, if for all $e \in B$, $F(e) = U$.

**Definition 2.9.** [6] Let $U$ be an initial universe and $E$ be a set of parameters. Let $\tau$ be a subcollection of $S(U)$, the collection of soft sets defined on $U$. Then $\tau$ is a soft topology if it satisfies the following conditions.

(i) $F_\emptyset, F_E \in \tau$.

(ii) The union of any number of soft sets in $\tau$ belongs to $\tau$. 
On soft $\beta$—open sets

(iii) The intersection of any two soft sets in $\tau$ belongs to $\tau$.

If this definition is considered for further development, then the results are similar to that of results in topological spaces. Therefore, throughout the paper the following definition of soft topology is used.

**Definition 2.10.** [9] Let $U$ be an initial universe and $E$ be a set of parameters. Let $F_A \in S(U)$. A soft topology on $F_A$, denoted by $\tau$, is a collection of soft subsets of $F_A$ having the following properties:

1. $F_A, F_\emptyset \in \tau$
2. $\{F_A \subseteq F_A : i \in I\} \subseteq \tau \Rightarrow \bigcup_{i \in I} F_A \in \tau$
3. $\{F_A \subseteq F_A : 1 \leq i \leq n, n \in \mathbb{N}\} \subseteq \tau \Rightarrow \bigcap_{i=1}^{n} F_A \in \tau$.

The pair $(F_A, \tau)$ is called a soft topological space.

The definition of soft closed sets is taken as in [9], then Example 2.1 in [17] gives a soft topological space in $(F_A, \tau)$ which there is no soft closed set. Therefore, we follow the following proper perfect definition of soft closed in soft topological spaces $(F_A, \tau)$.

**Definition 2.11.** [8] Let $(F_A, \tau)$ be a soft topological space in $F_A$. Elements of $\tau$ are called soft open sets. A soft set $F_B$ in $F_A$ is said to be a soft closed set in $F_A$, if its relative complement, $F_B^C \cap F_A$ belongs to $\tau$.

**Definition 2.12.** [16] Let $(F_A, \tau)$ be a soft topological space and $F_B$ be a soft set in $F_A$.

(i) The soft interior of $F_B$ is the soft set $\text{int}(F_B) = \bigcup\{F_C : F_C$ is soft open set and $F_C \subseteq F_B\}$.

(ii) The soft closure of $F_B$ is the soft set $\text{cl}(F_B) = \bigcap\{F_C : F_C$ is soft closed set and $F_B \subseteq F_C\}$.

Its clearly $\text{int}(F_B)$ is the largest soft open set contained in $F_B$ and $\text{cl}(F_B)$ is the smallest soft closed set containing $F_B$.

**Lemma 2.13.** [16] Let $(F_A, \tau)$ be a soft topological space and $F_B$ and $F_C$ be a soft subset of $F_A$. Then the following hold.

(i) $\text{int}(\text{int}(F_B)) = \text{int}(F_B)$,

(ii) $F_B \subseteq F_C$ implies $\text{int}(F_B) \subseteq \text{int}(F_C)$.

(iii) $\text{int}(F_B \cap F_C) = \text{int}(F_B \cap F_C)$.

(iv) $\text{int}(F_B \cup F_C) \subseteq \text{int}(F_B \cup F_C)$.

(v) $F_B$ is soft open set if and only if $F_B = \text{int}(F_B)$. 

Lemma 2.14. [16] Let \((F_A, \tilde{\tau})\) be a soft topological space and \(F_B\) and \(F_C\) be a soft subset of \(F_A\). Then the following hold.

(i) \(\text{cl}(\text{cl}(F_B)) = \text{cl}(F_B)\).

(ii) \(F_B \subseteq F_C\) implies \(\text{cl}(F_B) \subseteq \text{cl}(F_C)\).

(iii) \(\text{cl}(F_B) \cap \text{cl}(F_C) \subseteq \text{cl}(F_B \cap F_C)\).

(iv) \(\text{cl}(F_B) \cup \text{cl}(F_C) = \text{cl}(F_B \cup F_C)\).

(v) \(F_B\) is a soft closed set if and only if \(F_B = \text{cl}(F_B)\).

Lemma 2.15. [8] Arbitrary union of soft open sets is soft open and finite intersection of soft closed sets is soft closed.

Proposition 2.16. [16] Let \((F_A, \tilde{\tau})\) be a soft topological space over \(F_A\). Then the following hold.

(i) \(F_\emptyset, F_E\) are soft closed sets in \(F_A\)

(ii) The union of any two soft closed sets is a soft closed set in \(F_A\).

(iii) The intersection of any two soft closed sets is a soft closed set in \(F_A\).

Proposition 2.17. [16] If \(\{F_{B_\alpha}\}_{\alpha \in I}\) is a collection of soft sets, then the following hold.

(i) \(\text{\tilde{\cup}} \text{int}(F_{B_\alpha}) \subseteq \text{int}(\text{\tilde{\cup}} F_{B_\alpha})\).

(ii) \(\text{\tilde{\cup}} \text{cl}(F_{B_\alpha}) \subseteq \text{cl}(\text{\tilde{\cup}} F_{B_\alpha})\).

Lemma 2.18. [6]

(i) For every soft open set \(F_B\) in a soft topological space \((F_A, \tilde{\tau})\) and every soft set \(F_C\), we have \(\text{cl}(F_C) \cap F_B \subseteq \text{cl}(F_C \cap F_B)\).

(ii) For every soft closed set \(F_B\) in a soft topological space \((F_A, \tilde{\tau})\) and every soft set \(F_C\), we have \(\text{int}(F_B) \cup F_C \subseteq \text{int}(F_B \cup F_C)\).

Definition 2.19. [6] Let \(F_B\) be a soft subset of \(F_A\) of a soft topological space \((F_A, \tilde{\tau})\). \(F_B\) is a soft \(\alpha\)-open set, if \(F_B \subseteq \text{int}(\text{cl}(F_B))\) and a soft \(\alpha\)-closed set, if \(\text{cl}(\text{int}(F_B)) \subseteq F_B\).

Definition 2.20. [7] Let \(F_B\) be a soft subset of \(F_A\) of a soft topological space \((F_A, \tilde{\tau})\). \(F_B\) is a soft semi open set, if \(F_B \subseteq \text{cl}(\text{int}(F_B))\) and a soft semi closed set, if \(\text{int}(\text{cl}(F_B)) \subseteq F_B\).

Definition 2.21. [8] Let \(F_B\) be a soft subset of \(F_A\) of a soft topological space \((F_A, \tilde{\tau})\). \(F_B\) is a soft pre-open set, if \(F_B \subseteq \text{int}(\text{cl}(F_B))\) and a soft pre-closed set, if \(\text{cl}(\text{int}(F_B)) \subseteq F_B\).
3. Soft $\beta$-open set and Soft $\beta$-closed set

**Definition 3.1.** Let $F_B$ be a soft subset of $F_A$ of a soft topological space $(F_A, \tau)$. $F_B$ is a soft $\beta$-open set (or) a soft semi-pre open set, if $F_B \subseteq \text{cl}(\text{int}(F_B))$. The complement of a soft $\beta$-open set is a soft $\beta$-closed set.

**Theorem 3.2.** Let $F_B$ be a soft subset of $F_A$ be a soft topological space of $(F_A, \tau)$. Then $F_B$ is a soft $\beta$-closed set if only if $\text{int}(\text{cl}(F_B)) \subseteq F_B$.

**Proof.** Let $F_B$ be a soft $\beta$-closed set. Then $F_B^c$ is a soft $\beta$-open set and so $F_B^c \subseteq \text{cl}(\text{int}(F_B^c))$. Therefore, $F_B^c \subseteq \text{cl}(\text{int}(F_B))$ and so $F_B \subseteq \text{cl}(\text{int}(F_B))$. Conversely, let $\text{int}(\text{cl}(F_B)) \subseteq F_B$. Then $\text{int}(\text{cl}(F_B)) \subseteq F_B^c$ and so $\text{cl}(\text{int}(F_B)) \subseteq F_B^c$. Therefore, $F_B$ is a soft $\beta$-closed set. ■

**Example 3.3.** Let $U = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\} \subseteq E$.

$F_A = \{(e_1, \{a, b, c\}), (e_1, \{a, b\}), (e_1, \{a\})\}$.

Then $\text{int}(\text{cl}(F_B)) = F_{B_1}$, and $\text{int}(\text{cl}(F_B)) = F_{B_2}$, and $\text{int}(\text{cl}(F_B)) = F_{B_3}$, and $\text{int}(\text{cl}(F_B)) = F_{B_4}$, and $\text{int}(\text{cl}(F_B)) = F_{B_5}$, and $\text{int}(\text{cl}(F_B)) = F_{B_6}$, and $\text{int}(\text{cl}(F_B)) = F_{B_7}$, and $\text{int}(\text{cl}(F_B)) = F_{B_8}$, and $\text{int}(\text{cl}(F_B)) = F_{B_9}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{10}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{11}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{12}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{13}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{14}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{15}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{16}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{17}}$, and $\text{int}(\text{cl}(F_B)) = F_{B_{18}}$.

(i) Let us take $F_B = \{(e_1, \{b\}), (e_2, \{c\})\}$. Then $\text{int}(F_B) = F_{B_1}$, $\text{int}(F_B) = F_{B_2}$, $\text{int}(F_B) = F_{B_3}$, $\text{int}(F_B) = F_{B_4}$, $\text{int}(F_B) = F_{B_5}$, $\text{int}(F_B) = F_{B_6}$, $\text{int}(F_B) = F_{B_7}$, $\text{int}(F_B) = F_{B_8}$, $\text{int}(F_B) = F_{B_9}$, and $\text{int}(F_B) = F_{B_{10}}$. Hence $F_B$ is a soft semi-open set but not a soft $\alpha$-open set.

(ii) Let us take $F_B = \{(e_1, \{a\}), (e_2, \{a\})\}$. Then $\text{cl}(F_B) = F_{B_{11}}$, $\text{cl}(F_B) = F_{B_{12}}$, $\text{cl}(F_B) = F_{B_{13}}$, $\text{cl}(F_B) = F_{B_{14}}$, $\text{cl}(F_B) = F_{B_{15}}$, $\text{cl}(F_B) = F_{B_{16}}$, $\text{cl}(F_B) = F_{B_{17}}$, $\text{cl}(F_B) = F_{B_{18}}$. Hence $F_B$ is a soft pre-open set but not a soft $\alpha$-open set.

(iii) Let us take $F_B = \{(e_1, \{c\}), (e_2, \{b, c\})\}$. Then $\text{cl}(F_B) = F_{B_{19}}$, $\text{cl}(F_B) = F_{B_{20}}$, $\text{cl}(F_B) = F_{B_{21}}$, $\text{cl}(F_B) = F_{B_{22}}$, $\text{cl}(F_B) = F_{B_{23}}$, $\text{cl}(F_B) = F_{B_{24}}$, $\text{cl}(F_B) = F_{B_{25}}$, $\text{cl}(F_B) = F_{B_{26}}$. Hence $F_B$ is a soft semi-pre-open set but not a soft semi-pre-open set.

(iv) Let us take $F_B = \{(e_1, \{a, b\}), (e_2, \{c\})\}$. Then $\text{cl}(F_B) = F_{B_{27}}$, $\text{cl}(F_B) = F_{B_{28}}$, $\text{cl}(F_B) = F_{B_{29}}$, $\text{cl}(F_B) = F_{B_{30}}$, $\text{cl}(F_B) = F_{B_{31}}$, $\text{cl}(F_B) = F_{B_{32}}$, $\text{cl}(F_B) = F_{B_{33}}$, $\text{cl}(F_B) = F_{B_{34}}$. Hence $F_B$ is a soft semi-pre-open set but not a soft semi-pre-open set.

(v) Let us take $F_B = \{(e_1, \{a, b\}), (e_2, \{a\})\}$. Then $\text{cl}(F_B) = F_{B_{35}}$, $\text{cl}(F_B) = F_{B_{36}}$, $\text{cl}(F_B) = F_{B_{37}}$, $\text{cl}(F_B) = F_{B_{38}}$, $\text{cl}(F_B) = F_{B_{39}}$, $\text{cl}(F_B) = F_{B_{40}}$, $\text{cl}(F_B) = F_{B_{41}}$, $\text{cl}(F_B) = F_{B_{42}}$. Hence $F_B$ is a soft $\beta$-open set but it is neither a soft semi-open set nor a soft pre-open set.
4. Properties of soft $\beta$-open set and Soft $\beta$-closed set

Theorem 4.1. In a soft topological space $(F_A, \tau)$, we have the following conditions.

(i) $F_\phi, F_A$ are soft $\beta$-open sets.

(ii) The union of any number of soft $\beta$-open subsets is a soft $\beta$-open set.

Proof.

(i) $F_\phi$ and $F_A$ are open and so are soft $\beta$-open sets.

(ii) Let $F_\alpha$ belongs to soft $\beta$-open set in $F_A$ for $\forall \alpha \in I$, where I is an index set. Then for all $\alpha \in I$, $F_\alpha \subseteq \text{cl}(\text{int}(\text{cl}(F_\alpha))) \Rightarrow \bigcup_{\alpha \in I} F_\alpha \subseteq \bigcup_{\alpha \in I} \text{cl}(\text{int}(\text{cl}(F_\alpha)))$ and $\text{cl}(\bigcup_{\alpha \in I} \text{int}(\text{cl}(F_\alpha))) \subseteq \text{cl}(\bigcup_{\alpha \in I} \text{cl}(F_\alpha)) \subseteq \text{cl}(\bigcup_{\alpha \in I} \text{int}(\text{cl}(F_\alpha)))$. Therefore, union of any number of soft $\beta$-open subsets is a soft $\beta$-open set.

The intersection of two soft $\beta$-open sets need not be soft $\beta$-open set as is illustrated by the following example.

Remark 4.2. Let $(F_A, \tau)$ be the soft topological space of Example 3.3. We take $F_B = \{(e_1, \{b\}), (e_2, \{b\})\}$, then $\text{cl}(F_B) = F_{12}$, $\text{int}(\text{cl}(F_B)) = F_9$, and $\text{cl}(\text{int}(\text{cl}(F_B))) = F_{12}$, and so $F_B \subseteq \text{cl}(\text{int}(\text{cl}(F_B)))$. Hence $F_B$ is a soft $\beta$-open set and $F_C = \{(e_1, \{b\}), (e_2, \{a, c\})\}$, then $\text{cl}(F_C) = F_{14}$, $\text{int}(\text{cl}(F_C)) = F_2$, and $\text{cl}(\text{int}(\text{cl}(F_C))) = F_{14}$, and so $F_C \subseteq \text{cl}(\text{int}(\text{cl}(F_C)))$. Hence $F_C$ is a soft $\beta$-open set. Now, let $F_B \cap F_C = \{(e_1, \{b\}), (e_2, \{b\})\} \cap \{(e_1, \{b\}), (e_2, \{a, c\})\} = \{(e_1, \{b\})\}$. If $F_D = \{(e_1, \{b\})\}$, then $\text{cl}(F_D) = F_{16}$, $\text{int}(\text{cl}(F_D)) = F_\phi$, and $\text{cl}(\text{int}(\text{cl}(F_D))) = F_\phi$, and so $F_D \subseteq \text{cl}(\text{int}(\text{cl}(F_D)))$. Hence $F_D$ is not a soft $\beta$-open set.

Theorem 4.3. Any soft topological space $(F_A, \tau)$, arbitrary intersection of soft $\beta$-closed sets is a soft $\beta$-closed set.

Proof. Let $F_\alpha$ be a soft $\beta$-closed set in $F_A$ for every $\forall \alpha \in I$, where I is an index set. Then for every $\forall \alpha \in I$, $F_\alpha \supseteq \text{int}(\text{cl}(F_\alpha))) \Rightarrow \bigcap_{\alpha \in I} F_\alpha \supseteq \bigcap_{\alpha \in I} \text{int}(\text{cl}(F_\alpha)))$. Hence $F_B \cap F_C$ is soft $\beta$-closed set. The union of two soft $\beta$-closed sets need not be soft $\beta$-closed set as is illustrated by the following example.

Remark 4.4. Let $(F_A, \tau)$ be the soft topological space of Example 3.3. If $F_B, F_C$ are any two soft $\beta$-closed sets in $(F_A, \tau)$, then the following example shows that $F_B \cap F_C$ need not be a soft $\beta$-closed set.

We take $F_B = \{(e_1, \{a, b, c\}), (e_2, \{b\})\}$, then $\text{int}(F_B) = F_5$, $\text{cl}(\text{int}(F_B)) = F_{11}$, and $\text{cl}(\text{int}(\text{cl}(F_B))) = F_5$. Hence $F_B$ is a soft $\beta$-closed set, and $F_C = \{(e_1, \{a, b, c\}), (e_2, \{c\})\}$, then $\text{int}(F_C) = F_3$, $\text{cl}(\text{int}(F_C)) = F_{18}$, and $\text{cl}(\text{int}(\text{cl}(F_C))) = F_3$, and so $\text{cl}(\text{int}(\text{cl}(F_C))) \subseteq F_C$. Hence $F_C$ is a soft $\beta$-closed set. $F_B$ and $F_C$ are soft
Theorem 4.7. In an indiscrete soft topological space \((A, \tau)\) the following statements are true:

\[ \beta \text{-closed sets but } F_B \supseteq F_C = \{ (e_1, \{ a, b, c \}), (e_2, \{ b \}) \} \supseteq \{ (e_1, \{ a, b, c \}), (e_2, \{ c \}) \} \]

If \( F_D = \{ (e_1, \{ a, b, c \}), (e_2, \{ b, c \}) \} \), then \( \text{int}(F_D) = F_1 \)

\[ \text{cl}(\text{int}(F_D)) = F_A \text{ and so } \text{cl}(\text{int}(F_D)) \supseteq \text{cl}(F_D). \]

Hence \( F_D \) is not a soft \( \beta \)-closed set.

**Theorem 4.5.** If \( F_B \) is a soft \( \beta \)-open set in a soft topological space \((A, \tau)\) which is also a soft semi-closed set, then \( F_B \) is a soft semi-open set.

**Proof.** If \( F_B \) is soft \( \beta \)-open set, then \( F_B \subseteq \text{cl}(\text{int}(F_B)) \) and \( F_B \) is soft semi-closed set, then \( \text{int}(\text{cl}(F_B)) \subseteq F_B \)

and \( \text{cl}(\text{int}(F_B)) \subseteq F_B \). So \( \text{int}(\text{cl}(F_B)) \subseteq \text{cl}(\text{int}(F_B)) \).

If \( \text{cl}(F_B) = F_C \), then \( F_C \subseteq \text{cl}(F_C) \). Since \( F_C \) is a soft open set, \( F_B \) is a soft semi-open set.

**Corollary 4.6.** If \( F_B \) is a soft \( \beta \)-closed set in a soft topological space \((A, \tau)\) which is also a soft semi-closed set, then \( F_B \) is a soft semi-closed set.

**Theorem 4.7.** In an indiscrete soft topological space \((A, \tau)\), each soft \( \beta \)-open set is a soft pre-open set.

**Proof.** If \( F_B = F_\phi \), then \( F_B \) is a soft \( \beta \)-open set as well as a soft pre-open set. Let \( F_B \neq F_\phi \). If \( F_B \) is soft \( \beta \)-open, then \( F_B \subseteq \text{cl}(\text{int}(F_B)) \), since \( F_\phi \) and \( F_B \) are the only soft open sets. Hence \( F_B \) is a soft pre-open set.

**Theorem 4.8.** A soft subset \( F_B \) in a soft topological space \((A, \tau)\), is soft \( \beta \)-closed set if and only if \( \text{cl}(F_A - \text{cl}(\text{int}(F_B))) \)

\[ \supseteq (F_A - \text{cl}(\text{int}(F_B))) \supseteq \text{cl}(F_B) - F_B. \]

**Proof.** \( \text{cl}(F_A - \text{cl}(\text{int}(F_B))) \supseteq (F_A - \text{cl}(\text{int}(F_B))) \supseteq \text{cl}(F_B) - F_B \)

\[ \iff (F_A - \text{int}(\text{cl}(F_B))) \supseteq \text{cl}(F_B) - F_B \]

\[ \iff (F_A - \text{int}(\text{cl}(F_B))) \supseteq \text{cl}(F_B) - F_B \]

\[ \iff F_B \subseteq \text{int}(\text{cl}(F_B)) \iff F_B \text{ is a soft } \beta \text{-closed set.} \]

**Theorem 4.9.** If \( F_B \) is a soft \( \beta \)-open set in a soft topological space \((A, \tau)\) which is also a soft \( \alpha \)-closed set, then \( F_B \) is a soft closed set.

**Proof.** Let \( F_B \) be a soft \( \beta \)-open set in \( A \). Then \( F_B \subseteq \text{cl}(\text{int}(F_B)) \).

Since \( F_B \) is soft \( \alpha \)-closed set, then \( \text{cl}(\text{int}(F_B)) \subseteq F_B \), and so \( \text{cl}(\text{int}(F_B)) \subseteq F_B \subseteq \text{cl}(\text{int}(F_B)) \).

Hence \( F_B = \text{cl}(\text{int}(F_B)) \) which is soft closed.

**Definition 4.10.** Let \((A, \tau)\) be a soft topological space and let \( F_B \) be a soft subset of \( A \).

(i) The soft \( \beta \)-interior of \( F_B \) is the soft set \( \beta \text{-int}(F_B) = \bigcup \{ F_C : F_C \text{ is soft } \beta \text{-open set and } F_C \subseteq F_B \} \).

(ii) The soft \( \beta \)-closure of \( F_B \) is the soft set \( \beta \text{-cl}(F_B) = \bigcap \{ F_C : F_C \text{ is soft } \beta \text{-closed set and } F_B \subseteq F_C \} \).
Clearly, $\beta\text{-cl}(F_B)$ is the smallest soft $\beta$-closed set in $F_A$ which contains $F_B$ and $\beta\text{-int}(F_B)$ is the largest soft $\beta$-open set in $F_A$ which is contained in $F_B$.

**Theorem 4.11.** Let $(F_A, \tilde{\tau})$ be a soft topological space and let $F_B$ be a soft subset of $F_A$. Then the following hold.

(i) $F_B$ is soft $\beta$-closed if and only if $F_B = \beta\text{-cl}(F_B)$.

(ii) $F_B$ is soft $\beta$-open set if and only if $F_B = \beta\text{-int}(F_B)$.

**Proof.**

(i) Let $F_B = \beta\text{-cl}(F_B) = \bigcap \{F_C : F_C$ is soft $\beta$-closed set and $F_B \subseteq F_C\}$. This shows that $F_B \in \{F_C : F_C$ is a soft $\beta$-closed set and $F_B \subseteq F_C\}$. Hence $F_B$ is a soft $\beta$-closed set. Conversely, let $F_B$ be a soft $\beta$-closed set. Since $F_B \subseteq F_B$ and $F_B$ is a soft $\beta$-closed set, $F_B \in \{F_C : F_C$ is a soft $\beta$-closed set and $F_B \subseteq F_C\}$. $F_B = \bigcap \{F_C : F_C$ is soft $\beta$-closed set and $F_B \subseteq F_C\}$.

(ii) Let $F_B = \beta\text{-int}(F_B) = \bigcup \{F_C : F_C$ is soft $\beta$-open set and $F_C \subseteq F_B\}$. This shows that $F_B \in \{F_C : F_C$ is a soft $\beta$-open set and $F_C \subseteq F_B\}$. Hence $F_B$ is a soft $\beta$-open set. Conversely, let $F_B$ be a soft $\beta$-open set. Since $F_B \subseteq F_B$ and $F_B$ is a soft $\beta$-open set, $F_B \in \{F_C : F_C$ is a soft $\beta$-open set and $F_C \subseteq F_B\}$. Further, $F_B \subseteq F_C$ for all such that $F_B$, $F_B = \bigcap \{F_C : F_C$ is soft $\beta$-open set and $F_C \subseteq F_B\}$.

**Theorem 4.12.** Let $(F_A, \tilde{\tau})$ be a soft topological space and let $F_B$ and $F_C$ be two soft subsets of $F_A$. Then the following hold.

(i) $\beta\text{-cl}(F_\phi) = F_\phi$ and $\beta\text{-cl}(F_A) = F_A$.

(ii) $\beta\text{-cl}(\beta\text{-cl}(F_B)) = \beta\text{-cl}(F_B)$.

(iii) $F_B \subseteq F_C \Rightarrow \beta\text{-cl}(F_B) \subseteq \beta\text{-cl}(F_C)$.

(iv) $\beta\text{-cl}(F_B \cap F_C) = \beta\text{-cl}(F_B) \cap \beta\text{-cl}(F_C)$.

(v) $\beta\text{-cl}(F_B \cap F_C) \subseteq \beta\text{-cl}(F_B) \cap \beta\text{-cl}(F_C)$.

(vi) $(\beta\text{-cl}(F_B))^c = \beta\text{-int}(F_B)^c$.

**Proof.** Let $(F_A, \tilde{\tau})$ be a soft topological space and let $F_B$ be a soft subset of $F_A$.

(i) Since $F_\phi$ and $F_A$ are soft $\beta$-closed sets, by Theorem 3.14 (i), $\beta\text{-cl}(F_\phi) = F_\phi$ and $\beta\text{-cl}(F_A) = F_A$.

(ii) $\beta\text{-cl}(F_B)$ belongs to soft $\beta$-closed set in $F_A$ by definition of soft closure and [Theorem 3.5], $\beta\text{-cl}(\beta\text{-cl}(F_B)) = \beta\text{-cl}(F_B)$ by Theorem 3.14 (i).
Theorem 4.13. Let \((F_A, \tau)\) be a soft topological space and \(F_B\) be a soft subset of \(F_A\). Then the following hold.

(i) \(\beta\text{-int}(F_\emptyset) = F_\emptyset\) and \(\beta\text{-int}(F_A) = F_A\).

(ii) \(\beta\text{-int}(\beta\text{-int}(F_B)) = \beta\text{-int}(F_B)\).

(iii) \(F_B \subseteq F_C \Rightarrow \beta\text{-int}(F_B) \subseteq \beta\text{-int}(F_C)\).

(iv) \(\beta\text{-int}(F_B \cap F_C) = \beta\text{-int}(F_B) \cap \beta\text{-int}(F_C)\).

(v) \(\beta\text{-int}(F_B \cup F_C) = \beta\text{-int}(F_B) \cup \beta\text{-int}(F_C)\).

\begin{proof}
Let \((F_A, \tau)\) be a soft topological space and \(F_B\) and \(F_C\) be two soft subsets of \(F_A\).

(i) Since \(F_\emptyset\) and \(F_A\) are soft \(\beta\)-open sets, by Theorem 3.14 (ii), \(\beta\text{-int}(F_\emptyset) = F_\emptyset\) and \(\beta\text{-int}(F_A) = F_A\).

(ii) \(\beta\text{-int}(F_B)\) is a soft \(\beta\)-open set in \(F_A\) by definition of soft interior and [Theorem 3.4] and so \(\beta\text{-int}(\beta\text{-int}(F_B)) = \beta\text{-int}(F_B)\) by Theorem 3.14 (ii).
\end{proof}
Theorem 4.14. Let (\(F_A, \tau\)) be a soft tolopogical space. Then the following hold.

(i) \(\bar{\tau}\) soft \(\beta\)-open set in \((F_A, \tau)\).

(ii) If \(F_B\) is a soft subset of \(F_A\) and \(F_C\) is a soft pre-open set of \(F_A\) such that \(F_C \subseteq F_B \subseteq \text{cl}(\text{int}(F_C))\), then \(F_B\) is a soft \(\beta\)-open set.

Proof.

(i) It is obvious.

(ii) Since \(F_C\) is a soft pre-open set, we have \(F_C \subseteq \text{int}(\text{cl}(F_C))\). Then
\[
\text{cl}(\text{int}(\text{cl}(F_C))) = \text{cl}(\text{int}(F_C)) \subseteq \text{cl}(\text{int}(\text{cl}(F_B))),
\]
so \(F_B\) is a soft \(\beta\)-open set.

Theorem 4.15. Let \(F_B\) be a soft subset of \(F_A\) in a soft topological space \((F_A, \tau)\). Then the following hold.

(i) \(\alpha \cdot \text{cl}(F_B) = F_B \cdot \text{cl}(\text{int}(F_B))\).

(ii) \(\alpha \cdot \text{int}(F_B) = F_B \cdot \text{int}(\text{cl}(F_B))\).
Proof.

(i) \( \text{cl} (\text{int} (\text{cl}(F_B)) \cup \text{cl} (\text{int} (\text{cl}(F_B)))) \subseteq \text{cl} (\text{int} (\text{cl}(F_B))) \subseteq F_B \cup \text{cl} (\text{int} (\text{cl}(F_B))) \) which implies that \( F_B \cup \text{cl} (\text{int} (\text{cl}(F_B))) \) is a soft \( \alpha \)-closed set which implies that \( \text{cl} (\text{int} (\text{cl}(F_B))) \subseteq F_B \cup \text{cl} (\text{int} (\text{cl}(F_B))) \). Conversely, \( \text{cl} (\text{int} (\text{cl}(F_B))) \) is a soft \( \alpha \)-closed set, which implies that \( \text{cl} (\text{int} (\text{cl}(F_B))) \subseteq F_B \cup \text{cl} (\text{int} (\text{cl}(F_B))) \) which implies that \( F_B \cup \text{cl} (\text{int} (\text{cl}(F_B))) \subseteq \text{cl} (\text{int} (\text{cl}(F_B))) \) which implies that \( \text{cl} (\text{int} (\text{cl}(F_B))) = F_B \cup \text{cl} (\text{int} (\text{cl}(F_B))) \).

(ii) \( \text{int} (\text{cl} (\text{int} (\text{cl}(F_B)))) \subseteq \text{int} (\text{cl} (\text{int} (\text{cl}(F_B)))) \subseteq \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \) which implies that \( \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \subseteq \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \) which implies that \( \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \subseteq \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \) which implies that \( \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \subseteq \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \) which implies that \( \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \subseteq \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \) which implies that \( \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \subseteq \text{int} (\text{cl} (\text{int}(\text{cl}(\text{int}(F_B))))) \).

\( \square \)

5. Conclusion

We studied in this paper about Soft \( \beta \)-open sets and soft \( \beta \)-closed sets over the soft topological space. We also defined some properties of soft \( \beta \)-closed sets and soft \( \beta \)-open sets and examples where it can be applied were also described. Examples with details and several applications related to soft \( \beta \)-open sets and soft \( \beta \)-closed sets were dealt in detail.

References


