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Problems Of The Complex Idempotents In Group Ring C[G]

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Abstract

This research paper is on the concept of the complex idempotent in C[G]. Here C[G] is the group ring of group G over the complex numbers C. To find the idea of the complex idempotent in C[G] we have used the concept of Hermitian inner product on C[G], norms in C[G] as well as the absolute value of elements of C[G]. We have taken idempotent ***e*** in the complex group rings C[G] and proved that *tre* ***e*** ≥ 0. When group G is finite and K is a field of char K = 0 then 0≤ *tre* ***e*** ≤ 1 as 0≤ dim *V**e*** ≤|G|, and also *tre* ***e*** is always in prime subfield of K. There is a basic difference in these two properties of *tre* ***e***.

i. When *tre* ***e*** is contained in the prime subfield of K then it is an algebraic property.
ii. When the inequality 0≤ *tre* ***e*** ≤ 1, then it is an analytic property.

Thus we have obtained the analytic assertion 0≤ *tre* ***e*** ≤ 1 and the algebraic assertion on values of *tre* ***e*.** Such concept, when char of field K is p>0 is still a problem, of research.

Keywords: Complex idempotent, Hermitian inner product, Norms in C[G], Algebraic property, Analytic property of *tre* ***e***.

1. Introduction

This paper presents an idea of complex idempotent in in C[G]. For a finite group G and group ring K[G] we choose a homomorphism *f* such that, *f*: K[G] → M_n(K), here M_n(K) is a matrix of dimension *n* and *n* = dim *V* =|G|. *f*(*x*) is a permutation matrix of 0s and 1s having one in each row and column. If *x*≠1 then for all *x*∈G, *tre f*(*x*) = 0, but when *x* = 1 then *f*(*x*) is an identity matrix and *tre f*(1) =*n*=|G|. Let us suppose that $\alpha = \sum a_x.x \in$ K[G], then *tre f*(α) = $\sum a_x$.*tre f*(*x*) = $a_{1.}|G|$. Similarly, for an arbitrary G we define a map, *tre* K[G] →K and so, *tre* ($\sum a_x.x$)= a_1. Now we take some lemmas for basic results.

Lemma1: Let *tre*: K[G] →K be K-linear then for all $\alpha,\beta \in$ K[G] we have $tre\ \alpha\beta = tre\ \beta\alpha$.

Lemma2: Let us suppose that G be a finite group and *char* K is not divisor of O|G|, then

i. If $\alpha \in$ K[G] is nilpotent ⇒ *tre* α=0.
ii. If ***e***∈ K[G] is an idempotent ⇒ *tre* ***e*** = (dim K[G].***e***/|G|

2. Complex Group Ring C[G].

Let us suppose that G be an arbitrary group and C[G] be the group ring of G over the complex numbers C. Let α,β are elements in C[G], and $\alpha = \sum a_x.x$, $\beta = \sum b_x.x$, then the product [1][2] and norms [1] in C[G] as follow, $(\alpha, \beta) = \sum a_x \bar{b}_x$ $\|\alpha\| = (\alpha, \alpha)^{1/2} = (\sum |a_x|^2)^{1/2}$.

3. Let *e* be an Idempotent in C[G] and We Have to Prove *tre* *e* ≥ 0.

Lemma3: The Hermitian inner product [5] on C[G] will be, if $\alpha, \beta \in$C[G], (α, β)= *tre* $\bar{\beta}\alpha$ = *tre* $\alpha\bar{\beta}$. If a map $\alpha \to \bar{\alpha}$ is a ring anti-auto-morphism [3] of C[G] for all α,β,γ we have $(\alpha,\beta\gamma) = (\alpha\bar{\gamma},\beta) = (\bar{\beta}\alpha,\gamma)$

4. Decomposition of C[G] as Direct Sum of Two Right Ideals.

Let **I** =***e*** C[G] be the right ideal of C[G] generated by the idempotent ***e*** and let $\mathbf{I}^*$ be its orthogonal component. Since G is finite so C[G] is finite dimensional vector space. Hence **I**+ $\mathbf{I}^*$ **=** C[G] is a direct sum decomposition. Hence $\mathbf{I}^*$ is also a right ideal of C[G]. If If $\alpha \in$ **I** $\beta \in$ **I,*** and $\gamma \in$ C[G] then $\alpha\bar{\gamma} \in$ **I** (right ideal of C[G]. Hence, $(\alpha,\beta\gamma) = (\alpha\bar{\gamma},\beta)$= 0 and $\beta\gamma$ is orthogonal to all $\alpha \in$ **I,** also we get that $\beta\gamma \in \mathbf{I}^*$. Therefore, we have found that **I**+ $\mathbf{I}^*$ **=** C [G] is a decomposition of C[G] as a direct sum of two right ideals. Now we suppose that, $g + g^*$ =**I,** is similar decomposition of **I.** Thus g and g^* are idempotent with, **I** = g C[G] and **I** * = g^*C[G]. As g is orthogonal to g^*C[G] so g^*C[G] = (1-g)C[G] and for all α∈C[G] we have (g,(1-g)α) =($(\overline{1-g})$ g,α)=0 also $(\overline{1-g})\ g$ ∈C[G]* =0 ⇒$g = \bar{g}g$. Again $\bar{g} = \overline{\bar{g}g} = g\bar{g} = g$ and hence g is a self-adjoint [4] idempotent. Since, ***e***C[G] = **I**= gC[G], as ***e***, g are left identities of ideal **I**, hence, we have, *tre* ***e*** = *tre* g***e*** = *tre* ***e***g = *tre* g. But $g = \bar{g}g$ so, *tre* ***e*** = *tre* g = *tre* $g\bar{g} = \|g\|^2 \geq 0$. Therefore, *tre* ***e*** ≥0.

5. We have obtained that *tre* $\boldsymbol{e}$ ≥0 but still it remained to show that *tre* $\boldsymbol{e} \in \boldsymbol{Q}$, here $\boldsymbol{Q}$ is the set of rational numbers.

Since we get that **I**+ $\mathbf{I}^*$ = C[G] for finite, but *tre* $\boldsymbol{e}$ ∈Q is possible for infinite group G. Such decompositions are not true for infinite dimensional inner product spaces. So there are two ways to get this problem's solution.

i. First way was used by **Kaplansky (69)** [7] and **Montgomery (69)** [8].Both of them embedded C[G] in some larger algebra in which these decompositions are possible. These larger algebras were defined on suitable topologies on C[G].

ii. Second approach was given by **Passman (71)** [6]. His observations were based on element g, as taken above. Let us take $\alpha \in \mathbf{I}$ and distance between α and 1 ∈ C[G] as follow, d(α, 1)^2=$\|\alpha-1\|^2$=($\alpha-1$, $\alpha-1$). Since g+ g^* = 1 and ($\alpha-g$, g^*)=0. We have d(α, 1)^2= $\|\alpha-g\|^2$+$\|g^*\|$. So d(α, 1)≥ $\|g^*\|$, and becomes equal if only if α= g. Thus g is unique element of **I** and closest to 1. Now let G be an arbitrary group and L is a complex subspace of C[G] then , d(L, γ) = inf$\|\alpha-\gamma\|$. Hence, there exists a sequence of elements of **I** whose corresponding distances α ∈L approach to d(**I**, 1). This sequence plays the role of the element g.

Lemma4: If $\alpha, \beta \in$ L, a linear subspace of C[G], then

$|\beta, \alpha-\gamma|^2 \leq \|\beta\|^2 (\|\alpha-\gamma\|^2 - d(L,\gamma)^2)$

Lemma5: If α, β ∈C[G], then

i. $\|\alpha+\beta\| \leq \|\alpha\|+\|\beta\|$, $|\alpha+\beta| \leq |\alpha|+|\beta|$.

ii. $|tre\ \alpha| \leq \|\alpha\|$, $(\alpha, 1) = tre\ \alpha$

iii. $\|\alpha\beta\| \leq \|\alpha\|.|\beta|$, $|\alpha\beta| \leq |\alpha|.|\beta|$

6. When Idempotent of C[G] be $\boldsymbol{e}$≠0.

Now we suppose that $\boldsymbol{e}$≠0 be an idempotent in C[G] and take **I** = $\boldsymbol{e}$C[G]. Then **I** is a linear subspace [9] of C[G]. We take d = d(**I**, 1) be the distance between **I** and 1. For each integer n>0 we get f_n ∈**I** with $\|g_n-1\|^2 < d^2 + 1/n^4$

Lemma6: If there exist non-negative real constants r' and r'' then we have,

i. I$\|g_n\|^2$-$tre g_n$I≤ r'/n

ii. $\|g_n\boldsymbol{e}-\boldsymbol{e}\| \leq r''/n$

Lemma7: If the trace of *e* is real then, *tre e* $\geq \|\boldsymbol{e}\|^2/|\boldsymbol{e}|^2 > 0$.

Proof: From lemma 5(ii) and 6 we get,

I$\|g_n\|^2$-*tre* g_nI$\leq r'/n$ and $\|g_n\boldsymbol{e} - \boldsymbol{e}\| \leq r''/n$ so we have tre $g_n.\boldsymbol{e}$ = *tre* g_n. ***e***=*tre* g_n. Since $g_n \in \boldsymbol{e}$ C[G]$\Rightarrow$ $\boldsymbol{e}.g_n = g_n$. So we have from the above,

I$\|g_n\|^2$-*tre*g_nI$\leq r' + r''/n$ as well as *tre* ***e*** = $\lim_{n\to\infty}\|g_n\|^2$.Thus *tre* ***e*** is real and non-negative. Now from lemma 5 (i) (iii) and 6(ii) we get, $\|\boldsymbol{e}\| \leq \|\boldsymbol{e} - g_n\|$+$\|g_n.\boldsymbol{e}\| \leq r^n/n$+$\|g_n\|.|\boldsymbol{e}|$. By taking limit as $n\to\infty$ we obtain $\|\boldsymbol{e}\| \leq (tre\ \boldsymbol{e})^{1/2}.|e|$ and hence, *tre* ***e***$\geq \|\boldsymbol{e}\|^2/|\boldsymbol{e}|^2 > 0$

7. An Analytic Proof of *tre* *e* is Algebraic over *Q*.

Theorem 8 [Kaplansky (69)] [7]: Let K be a field of characteristic 0 and let ***e*** $\neq$0,1 be an idempotent in K[G]. Then *tre* ***e*** is a totally real algebraic number with the property that, it and its algebraic conjugates lie strictly between 0 and 1.

Poof: Let us take ***e*** = $\sum b_x.x$ $\in$K[G] be an idempotent and F is finitely generated field extension of the rational number *Q*. It is given by F = *Q* (b_xIx $\in$ Supp ***e***). Thus ***e*** $\in$F[G] $\subset$C[G], and we suppose ***e*** as an element of C[G].Here ***e*** is an idempotent with same trace and ***e*** $\neq$ 0, 1. Now by lemma 7, we have *tre* ***e*** >0. As 1-***e*** is also a non-zero idempotent of C[G], so *tre* (1-***e***) = 1- *tre* **e**>0 and *tre* ***e*** <1. Now we suppose σ as any field automorphism of the complex numbers. Then σ includes a ring automorphism of C[G] by $\alpha = \sum a_x x \Rightarrow \alpha^\sigma = \sum a_x^\sigma x$. Since e^σ is again an idempotent of C[G], and *tre* $\boldsymbol{e}^{\boldsymbol{\sigma}}$=$(tre\ \boldsymbol{e})^\sigma$. Therefore we get 0< $(tre\ \boldsymbol{e})^\sigma$ <1 for all such σ. But if *tre* ***e*** were transcendental [10 over *Q*, then there would exist a field auto-morphism σ with $(tre\ \boldsymbol{e})^\sigma$ not real. It is a contradiction. Therefore, *tre* ***e*** is algebraic over *Q*. Hence, the other two idempotent of are 0 and 1 also their traces are 0 as well as 1.

8. Von Neumann Finite Ring

A ring R is said a von Neumann finite [11] if $\alpha.\beta$ =1 in R$\Rightarrow$$\beta.\alpha$=1.

Corollary 9: If K is a field of characteristic 0, then K[G] is von Neumann finite.

Proof: Let us take that $\alpha, \beta \in$ K[G] and $\alpha.\beta$=1. We take ***e*** = $\beta.\alpha$. Then $\boldsymbol{e}^2$= $\beta\alpha.\beta\alpha$= $\beta(\alpha\beta)\alpha$ = $\beta\alpha$=***e***. Therefore ***e*** is an idempotent in K[G]. But from lemma 1, *tre* ***e*** = *tre* $\beta\alpha$= *tre* $\alpha\beta$=1. Hence by theorem 8, we have ***e*** = 1.

9. What Are To Be Investigated Yet In Complex Idempotent?

These all conditions are problems of research, when the characteristic of field is greater than 0, in future.

i. If K is a field of characteristic p>0, then K[G] is von Neumann finite.
ii. In a group ring C[G], *tre* ***e*** ≥0.
iii. *tre* ***e*** is algebraic in Q.
iv. Analytic assertion 0≤ *tre* ***e*** ≤1.

Conclusions

In this research paper, we have presented the concept of complex idempotent in C[G] in easy way. For this purpose we have used Hermitian inner product as well as norm. With the help of these concept we get result as *tre* ***e*** ≥0. We have got that *tre* ***e*** is algebraic over *Q*. It is not transcendental over *Q*. Therefore, we obtained, the analytic assertion 0≤ *tre* ***e*** ≤1 and the algebraic assertion on values of *tre* ***e*** . This has been found when characteristic of field K is 0. But, when the characteristic of field K is p>0 then such concept of complex idempotent of C[G] is not known. It is still a matter of research in future.

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