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Necessary optimality conditions for a Lotka-Volterra three species system

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Abstract

An optimal control problem is studied for a Lotka-Volterra system of three differential equations. It models an ecosystem of three species which coexist. The species are supposed to be separated from each others. Mathematically, this is modeled with the aid of two control variables. Some necessary conditions of optimality are found in order to maximize the total number of individuals at the end of a given time interval.

Key Words and Phrases. Adjoint system, bang-bang control, cost functional, Pontrjagin's maximum principle, transversality conditions.

1. Introduction

We study the Lotka-Volterra three populations system of differential equations

$$\begin{cases} y_1' = y_1\left(a_1 - b_1 y_2 + c_1 y_3\right) \\ y_2' = y_2\left(-a_2 + b_2 y_1\right) \\ y_3' = y_3\left(a_3 - b_3 y_1\right), \end{cases} \tag{1.1}$$

where a_i, b_i, c_1 $(i = 1, 2, 3)$ are positive constants.

It is a nonlinear mathematical model of an ecosystem consisting of a herbivorous species (the number of individuals of which is y_1), a carnivorous one (y_2), and of plants, the quantity of which is denoted y_3.

The same model can also be applied to an ecosystem of a pest species (y_1), a predator (y_2), and a plant (y_3).

Obviously, for every initial value $(y_1(0), y_2(0), y_3(0))$, the above system has a unique solution (y_1, y_2, y_3) which is continuous on $[0, +\infty)$ ([7]). A first integral of the system is

$$(y_2(t))^{b_3}(y_3(t))^{b_2} e^{b_2 b_3 t} = C,$$

where C is a constant depending on the initial values ([7]):

$$C = (y_2(0))^{b_3}(y_3(0))^{b_2}.$$

It is easy to see that, supposing also the condition $\sigma = (a_2/b_2) - (a_3/b_3) > 0$, the components y_1, y_2 of the solution are bounded, both from above and from below:

$$\limsup_{t\to\infty} y_1(t) < +\infty,\ \liminf_{t\to\infty} y_1(t) > 0,\ \limsup_{t\to\infty} y_2(t) < +\infty,\ \liminf_{t\to\infty} y_2(t) > 0,$$

and as a consequence, $\limsup_{t\to\infty} y_3(t) < +\infty$ ([7]).

Observe that if $y_3 \equiv 0$, system (1.1) reduces to the well-known prey-predator system (see for example $[9]$). If $y_1 \equiv 0$, then the equations two and three become independent. In the absence of the herbivorous species, the quantity of plants grows exponentially, while the predator goes to extinction.

We now introduce in the ecosystem some control variables u and v, whose role is to separate (partially or totally) the three populations from each other. Denote by $1-u$ the rate of separation between the herbivorous and the carnivorous species y_1, y_2 and by $1-v$ the rate of separation between the herbivorous species and the plants. The functions u, v take values in the interval $[0,1]$. Then the controlled system is

$$\begin{cases} y_1' = y_1(a_1 - b_1 y_2 u + c_1 y_3 v) \\ y_2' = y_2(-a_2 + b_2 y_1 u) \\ y_3' = y_3(a_3 - b_3 y_1 v). \end{cases} \tag{1.2}$$

We study it on a finite time interval $[0,T]$. One imposes the initial conditions:

$$y_1(0) = y_1^0,\ y_2(0) = y_2^0,\ y_3(0) = y_3^0. \tag{1.3}$$

If $y_i^0 > 0$, $i = 1,2,3$, then the Cauchy problem $(1.2) - (1.3)$ has a unique positive solution, which is bounded on $[0,T]$ provided that $\sigma > 0$.

When $u = 0$, the rate of separation between the herbivorous (y_1) and carnivorous (y_2) species is 1, so that y_1 and y_2 are completely separated, that is they do not interact. In this case, the first equation in the system is independent of y_2, while the second equation does not depend on y_1. When $u = 1$, it follows that the rate of separation is 0, so that y_1 and y_2 are not separated at all. These are two extreme situations. Generally, the interaction between y_1 and y_2 is controlled at least partially. Similarly for $v = 0$ and for $v = 1$.

We intend to maximize the total population at the end of the time interval $[0,T]$. Hence the functional we have to maximize is $y_1(T) + y_2(T) + y_3(T)$, i.e. the control problem is

$$Minimize\{-y_1(T) - y_2(T) - y_3(T)\}, \tag{1.4}$$

where the control (u, v) belongs to the set

$$\mathcal{U} = \{(u, v) : [0, T] \to \mathbb{R}^2,\ u, v \text{ measurable},$$

$$0 \leq u(t) \leq 1, 0 \leq v(t) \leq 1, \text{ a.e. on } [0, T]\}, \tag{1.5}$$

while (y_1, y_2, y_3) is the solution of the Cauchy problem $(1.2) - (1.3)$.

The boundedness of y_1, y_2, y_3 allows us to use Theorem 1.2, pp. 43 in V. Barbu's book ($[5]$), to obtain the existence of an optimal control (u, v) in the class of bounded measurable functions.

In Section 2, Pontrjagin's maximum principle and its attendant transversality condition are invoked to find the optimal control (u, v). We deduce that u and v are bang-bang control and their values depend on the signs of the functions $-b_1p_1+b_2p_2$ and $c_1p_1-b_3p_3$ respectively, where (p_1, p_2, p_3) is the solution of the adjoint system. The discussion of these signs is the subject of Section 3. One establishes thus the final form of the optimal control (u, v), depending on the signs of $b_2 - b_1$ and $c_1 - b_3$. Section 4 contains some conclusions.

For a prey-predator system ($y_3 \equiv 0$), a similar problem was treated in $[10]$. In this case, we have a unique control variable $u : [0, T] \to \mathbb{R}$, $0 \leq u(t) \leq 1$ a.e. on $[0, T]$. If the sizes of the prey and predator populations depend also on their position in the habitat, the dynamics of the ecosystem is given by a nonlinear reaction-diffusion system. In $[4]$, one solves the problem of maximization of the total density of the two populations.

An optimal time problem for system (1.1) is the subject of $[7]$. A control variable u is introduced in the ecosystem, acting on y_1 as a chemical pesticide. Thus we have a double struggle against the herbivorous species: a biological one and a chemical one.

In $[6]$, the authors present several predator-prey PDE models and review recent results concerning the existence of positive steady-state solutions, of non-constants positive solutions, bifurcation, and so on. In $[8]$, a discrete prey-predator system is considered. The bifurcation theory is applied to show that the system can undergo fold, flip and Neimark-Sacker bifurcations. Necessary conditions and sufficient conditions for internal stabilizability of a Holling type II prey-predator system are given in $[2]$. Optimal control problems for age-structured population dynamics models are presented in $[1]$, $[3]$.

2. Necessary optimality conditions

We find some necessary optimality conditions in order to maximize the total number of individuals at the end of the given time interval $[0, T]$. In other words, we are lead to consider the following optimal control problem: find the control $(u, v) \in \mathcal{U}$ and the corresponding state (y_1, y_2, y_3) of the system $(1.2) - (1.3)$, which minimize the cost functional

$$\Phi(y_1, y_2, y_3, u, v) = -y_1(T) - y_2(T) - y_3(T). \tag{2.1}$$

To solve this problem, one uses Pontrjagin's maximum principle. Denote

$$y=\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix},\ f(y,u,v)=\begin{pmatrix} y_1(a_1-b_1y_2u+c_1y_3v)\\ y_2(-a_2+b_2y_1u)\\ y_3(a_3-b_3y_1v)\end{pmatrix},\ p=\begin{pmatrix} p_1\\ p_2\\ p_3\end{pmatrix}.$$

Here p is the adjoint variable, that is the solution of the associated adjoint system (2.3) below. The Hamiltonian function is

$$H(p,y,u,v)=p_1y_1(a_1-b_1y_2u+c_1y_3v)+$$
$$+p_2y_2(-a_2+b_2y_1u)+p_3y_3(a_3-b_3y_1v), \tag{2.2}$$

while the adjoint system is

$$\begin{cases} p_1'=-a_1p_1-y_2u(-b_1p_1+b_2p_2)-y_3v(c_1p_1-b_3p_3)\\ p_2'=a_2p_2-y_1u(-b_1p_1+b_2p_2)\\ p_3'=-a_3p_3-y_1v(c_1p_1-b_3p_3),\ t\in[0,T].\end{cases} \tag{2.3}$$

The transversality conditions are

$$p_1(T)=p_2(T)=p_3(T)=1. \tag{2.4}$$

Recall that, if H is a real Hilbert space with the scalar product $(.,.)$, then the normal cone to the closed and convex subset $K\subseteq H$ at the point a is defined by

$$N_K(a)=\{v\in H,\ (v,a-x)\geq 0,\ (\forall)\,x\in K\}.$$

If $N_{[0,1]^2}(u,v)$ is the normal cone to $[0,1]^2$ at the point (u,v) and $f_{u,v}^*$ is the adjoint of the Jacobian matrix $f_{u,v}\in L(\mathbb{R}^2,\mathbb{R}^2)$, the optimality condition

$$f_{u,v}^*(y,u,v)\cdot p(t)\in N_{[0,1]^2}(u,v)\ \text{ a.e on }\ (0,T),$$

(see for example $[5]$) becomes

$$\begin{pmatrix} -b_1y_1y_2 & b_2y_1y_2 & 0\\ c_1y_1y_3 & 0 & -b_3y_1y_3\end{pmatrix}\begin{pmatrix} p_1\\ p_2\\ p_3\end{pmatrix}\in N_{[0,1]^2}(u,v)\ \text{ a.e. on }\ (0,T). \tag{2.5}$$

But $N_{[0,1]^2}(u,v)=N_{[0,1]}(u)\times N_{[0,1]}(v)$ and

$$N_{[0,1]}(u)=\begin{cases} 0,\ u\in(0,1)\\ \mathbb{R}_-,\ u=0\\ \mathbb{R}_+,\ u=1.\end{cases}$$

Thus the inclusion (2.5) implies that

$$u(t)=\begin{cases} 0,\ \text{if } y_1y_2(-b_1p_1+b_2p_2)<0\\ 1,\ \text{if } y_1y_2(-b_1p_1+b_2p_2)>0\end{cases}$$

and

$$v(t)=\begin{cases}0, \text{ if } y_1y_3(c_1p_1-b_3p_3)<0\\ 1, \text{ if } y_1y_3(c_1p_1-b_3p_3)>0.\end{cases}$$

Since $y_i(t)>0,\ (\forall)\,t\in[0,T],\ (\forall)\,i=1,2,3,$ we get

$$u(t)=\begin{cases}0, \text{ if } -b_1p_1+b_2p_2<0, \text{ a.e. } t\in(0,T)\\ 1, \text{ if } -b_1p_1+b_2p_2>0, \text{ a.e. } t\in(0,T),\end{cases} \tag{2.6}$$

$$v(t)=\begin{cases}0, \text{ if } c_1p_1-b_3p_3<0, \text{ a.e. } t\in(0,T)\\ 1, \text{ if } c_1p_1-b_3p_3>0, \text{ a.e. } t\in(0,T).\end{cases} \tag{2.7}$$

In the next section, we discuss the form of the optimal control (u,v) according to the signs of $-b_1p_1+b_2p_2$ and $c_1p_1-b_3p_3$.

3. The form of the optimal control

We restrict ourselves to the case $a_3>a_1$. Observe that

$$\begin{cases}y_1u(-b_1p_1+b_2p_2)\geq 0,\ y_2u(-b_1p_1+b_2p_2)\geq 0,\\ y_1v(c_1p_1-b_3p_3)\geq 0,\ y_3v(c_1p_1-b_3p_3)\geq 0,\end{cases} \tag{3.1}$$

a.e. on $(0,T)$. By $(2.3)-(2.4)$, we can easily deduce that

$$p_1(t)=e^{a_1(T-t)}\{1+\int_t^T[y_2u(-b_1p_1+b_2p_2)+y_3v(c_1p_1-b_3p_3)](s)e^{-a_1(T-s)}ds\}$$

and analogously for p_2,p_3. Therefore,

$$p_1(t)\geq e^{a_1(T-t)}\geq 1,\ p_2(t)>0,\ p_3(t)\geq e^{a_3(T-t)}\geq 1,\ t\in(0,T). \tag{3.2}$$

By (2.3) it follows that p_1 and p_3 are nonincreasing a.e. on $[0,T]$. We have the following cases:

1. $-b_1+b_2<0, c_1-b_3<0$. In this case, $(-b_1p_1+b_2p_2)(T)<0$ and $(c_1p_1-b_3p_3)(T)<0$. Hence, there exists $\varepsilon>0$ such that both $-b_1p_1+b_2p_2<0$ and $c_1p_1-b_3p_3<0$ on $(T-\varepsilon,T]$. According to (2.6), (2.7), we have $u(t)=v(t)=0,\ (\forall)\,t\in(T-\varepsilon,T]$ and consequently, system $(2.3)-(2.4)$ becomes

$$\begin{cases}p_1'=-a_1p_1,\ p_2'=a_2p_2,\ p_3'=-a_3p_3,\ t\in(T-\varepsilon,T],\\ p_i(T)=1,\ i=1,2,3.\end{cases} \tag{3.3}$$

This yields

$$p_1(t)=e^{a_1(T-t)},\ p_2(t)=e^{-a_2(T-t)},\ p_3(t)=e^{a_3(T-t)},(\forall)\,t\in(T-\varepsilon,T].$$

Since $(-b_1p_1+b_2p_2)' = a_1b_1p_1 + a_2b_2p_2 > 0$ on $(T-\varepsilon, T]$, the inequality $-b_1p_1 + b_2p_2 < 0$ and the equality $u(t) = 0$ hold on the whole interval $[0,T]$. Similarly, for $a_3 > a_1$ and $c_1 < b_3$, $(c_1p_1 - b_3p_3)' > a_1c_1(p_3 - p_1) > 0$ on $(T-\varepsilon, T]$. Then, $c_1p_1 - b_3p_3 < 0$ holds on $[0,T]$ and $v(t) = 0$, $t \in [0,T]$.

The optimal state can be realized by system (1.2) for $u(t) = v(t) = 0$ on $[0,T]$:

$$y_1(t) = y_1^0 e^{a_1 t},\ y_2(t) = y_2^0 e^{-a_2 t},\ y_3(t) = y_3^0 e^{a_3 t},\ t \in [0,T]. \tag{3.4}$$

2. $-b_1 + b_2 < 0$, $c_1 = b_3$. As in the previous case, there exists $\varepsilon_1 > 0$ such that $-b_1p_1 + b_2p_2 < 0$ on $(T-\varepsilon_1, T]$, so $u(t) = 0$ on $(T-\varepsilon_1, T]$. The adjoint system has the form

$$\begin{cases} p_2(t) = e^{-a_2(T-t)} \\ p_1' = -a_1p_1 + b_3y_3v(-p_1+p_3) \\ p_3' = -a_3p_3 + b_3y_1v(-p_1+p_3). \end{cases} \tag{3.5}$$

Integrating the last two equations and using (2.4), one obtains

$$(c_1p_1 - b_3p_3)(t) = b_3(p_1 - p_3)(t) =$$

$$= -b_3 \int_t^T [a_3p_3 - a_1p_1 + b_3v(y_3 - y_1)(-p_1+p_3)](s)\,ds.$$

Hence $c_1p_1 - b_3p_3 < 0$ in a left neighborhood $(T-\varepsilon_2, T]$ of T. If we take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, then $-b_1p_1 + b_2p_2 < 0$ and $c_1p_1 - b_3p_3 < 0$ on $(T-\varepsilon, T)$. Arguing as in the first case, we see that $u(t) = v(t) = 0$ on $[0,T]$ and y_1, y_2, y_3 are given by (3.4).

3. $-b_1+b_2 < 0, c_1 - b_3 > 0$. By (2.4), $(-b_1p_1 + b_2p_2)(T) < 0$ and $(c_1p_1 - b_3p_3)(T) > 0$, so there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $-b_1p_1 + b_2p_2 < 0$ on $(T-\varepsilon_1, T]$ and $c_1p_1 - b_3p_3 > 0$ on $(T-\varepsilon_2, T]$. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. On $(T-\varepsilon, T]$ we have both $-b_1p_1 + b_2p_2 < 0$ and $c_1p_1 - b_3p_3 > 0$, therefore $u = 0$ and $v = 1$ on $(T-\varepsilon, T]$. The adjoint system (2.3) becomes

$$\begin{cases} p_2(t) = e^{-a_2(T-t)} \\ p_1' = -a_1p_1 - y_3(c_1p_1 - b_3p_3) \\ p_3' = -a_3p_3 - y_1(c_1p_1 - b_3p_3), \end{cases} \tag{3.6}$$

for $t \in (T-\varepsilon, T]$. Put

$$\theta = \inf\{t \in [0,T],\ (c_1p_1 - b_3p_3)(s) > 0,\ (\forall)\, s \in (t,T]\}. \tag{3.7}$$

If $\theta = 0$, then $v = 1$, $(\forall)\, t \in [0,T]$. In addition, since $-b_1p_1 + b_2p_2$ is monotonically increasing on $(T-\varepsilon, T]$, we have $(-b_1p_1 + b_2p_2)(t) < 0$, $t \in [0,T]$ and consequently $u(t) = 0$, $t \in [0,T]$.

If $\theta > 0$, then $(c_1p_1 - b_3p_3)(t) > 0$, $(\forall)\, t \in (\theta, T]$, $v(t) = 1$, $(\forall)\, t \in (\theta, T]$, and

$$c_1p_1(\theta) - b_3p_3(\theta) = 0. \tag{3.8}$$

But the function $-b_1p_1 + b_2p_2$ is monotonically increasing on $(T - \varepsilon, T]$; this implies that $(-b_1p_1 + b_2p_2)(t) < 0,\ t \in (T - \varepsilon, T]$, so $(-b_1p_1 + b_2p_2)(t) < 0$ holds on the whole $(\theta, T]$. Then $u(t) = 0$ on $(\theta, T]$ and the adjoint system (3.6) is applicable even on $(\theta, T]$. Repeating the above reasoning concerning the monotonicity of $-b_1p_1 + b_2p_2$ (using θ instead of T), we find that $-b_1p_1 + b_2p_2 < 0$ in a left neighborhood $(\theta - \varepsilon_3, \theta]$ of θ. The optimal control u is still 0 on $(\theta - \varepsilon_3, \theta]$.

To find v on $(\theta - \varepsilon_3, \theta]$, we study the monotonicity of $c_1p_1 - b_3p_3$. System (2.3) can be written as

$$\left\{ \begin{array}{c} p_1' = -a_1p_1 - y_3v\,(c_1p_1 - b_3p_3) \\ p_2' = a_2p_2 \\ p_3' = -a_3p_3 - y_1v\,(c_1p_1 - b_3p_3)\,, \end{array} \right. \tag{3.9}$$

for $t \in (\theta - \varepsilon_3, \theta]$. One obtains

$$(c_1p_1 - b_3p_3)' = -a_1c_1p_1 + a_3b_3p_3 +$$
$$+v\,(b_3p_3 - c_1p_1)\,(c_1y_3 - b_3y_1)\,,\ t \in (\theta - \varepsilon_3, \theta].$$

By (3.8) we get

$$(c_1p_1 - b_3p_3)\,(t) = \int_t^\theta (a_1c_1p_1 - a_3b_3p_3)\,(s)\,ds +$$
$$+\int_t^\theta v\,(s)\,(c_1p_1 - b_3p_3)\,(s)\,(c_1y_3 - b_3y_1)\,(s)\,ds,\ t \in (\theta - \varepsilon_3, \theta).$$

Since

$$(c_1p_1 - b_3p_3)\,(s) \to 0, (a_1c_1p_1 - a_3b_3p_3)\,(s) \to b_3\,(a_1 - a_3)\,p_3\,(\theta)\,,$$

as $s \to \theta - 0$, it follows that $c_1p_1 - b_3p_3 < 0,\ t \in (\theta - \varepsilon_3, \theta)$. Thus $v = 0$ for $t \in (\theta - \varepsilon_3, \theta]$.

The adjoint system has the form (3.3) on $(\theta - \varepsilon_3, \theta]$ and the first case can be applied to obtain that $u(t) = v(t) = 0,\ t \in [0, \theta]$. We conclude that $u = 0$ on $[0, T]$ and v admits a unique switching point $\theta \in (0, T)$, that is

$$v\,(t) = \left\{ \begin{array}{l} 0,\ t \in [0, \theta] \\ 1,\ t \in (\theta, T]. \end{array} \right. \tag{3.10}$$

Therefore, in case 3, $u = 0$ on $[0, T]$ and v has at most one switching point $\theta \in (0, T)$, which can be found from equation (3.8). If there is no switching point for v, then $v = 1$ on $[0, T]$. If there exists a unique switching point θ, then v is given by (3.10). We say that v is a bang-bang control.

4. $b_1 = b_2,\ c_1 - b_3 < 0$. Then, $(\exists)\,\varepsilon_2 > 0$ such that $(c_1p_1 - b_3p_3)\,(t) < 0,\ (\forall)\,t \in (T - \varepsilon_2, T]$. Thus, $v = 0$ on $(T - \varepsilon_2, T]$ and the adjoint system has the form

$$\left\{ \begin{array}{c} p_1' = -a_1p_1 - b_1y_2u\,(-p_1 + p_2) \\ p_2' = a_2p_2 - b_1y_1u\,(-p_1 + p_2) \\ p_3' = -a_3p_3, \end{array} \right. \tag{3.11}$$

for $t \in (T-\varepsilon_2, T]$. Since

$$(-b_1p_1 + b_2p_2)' = b_1[(a_1p_1 + a_2p_2) + b_1u\,(y_2 - y_1)\,(p_2 - p_1)],$$

we can easily see that $-b_1p_1 + b_2p_2 < 0$ in a small left neighborhood $(T-\varepsilon_1, T]$ of T. As in the first case, it follows that $u\,(t) = v\,(t) = 0$, $t \in [0, T]$ and the optimal state is given by (3.4) .

5. $b_1 = b_2$, $c_1 = b_3$. Integrating system $(2.3) - (2.4)$, we find

$$(-b_1p_1 + b_2p_2)\,(t) = b_1\,(-p_1 + p_2)\,(t) =$$

$$= b_1 \int_t^T [-a_1p_1 - a_2p_2 + b_1u\,(y_2 - y_1)\,(p_1 - p_2) - b_3y_3v\,(p_1 - p_3)]\,(s)\,ds.$$

Since $-a_1p_1 - a_2p_2 \to -a_1 - a_2$, $p_1 - p_2 \to 0$, and $p_1 - p_3 \to 0$ as $s \to T - 0$, there exits $\varepsilon_1 > 0$ such that $(-b_1p_1 + b_2p_2)\,(t) < 0$, $(\forall)\,t \in (T-\varepsilon_1, T]$. On this interval $u\,(t) = 0$.

Similarly,

$$(c_1p_1 - b_3p_3)\,(t) = b_3\,(p_1 - p_3)\,(t) =$$

$$= b_3 \int_t^T [a_1p_1 - a_3p_3 + b_3v\,(y_3 - y_1)\,(p_1 - p_3) - b_1y_2u\,(p_1 - p_2)]\,(s)\,ds$$

and therefore there is a neighborhood $(T-\varepsilon_2, T]$ of T, where $c_1p_1 - b_3p_3 < 0$ and thus $v = 0$. We continue like in the first case to conclude that $u = v = 0$ on $[0, T]$.

6. $b_1 = b_2$, $c_1 - b_3 > 0$. There exists $\varepsilon_2 > 0$ such that $c_1p_1 - b_3p_3 > 0$ on $(T-\varepsilon_2, T]$. Let θ be defined by the equality

$$\theta = \inf\{t \in [0, T]\,,\ (c_1p_1 - b_3p_3)\,(s) > 0, (\forall)\,s \in (t, T]\}. \tag{3.12}$$

Then $v\,(t) = 1$, $(\forall)\,t \in (\theta, T]$ and

$$(c_1p_1 - b_3p_3)\,(\theta) = 0. \tag{3.13}$$

From the adjoint system we deduce

$$(-b_1p_1 + b_2p_2)\,(t) = b_1\,(-p_1 + p_2)\,(t) =$$

$$= -b_1 \int_t^T [a_1p_1 + a_2p_2 + b_1u\,(y_1 - y_2)\,(p_1 - p_2) + y_3(c_1p_1 - b_3p_3)]\,(s)\,ds,$$

hence $-b_1p_1 + b_2p_2 < 0$ on a left neighborhood $(T-\varepsilon_1, T)$ of T and u is 0 on $(T-\varepsilon_1, T]$. One repeats the reasoning of the third case to conclude that $u = 0$ on $[0, T]$, while v admits at most one switching point. If such a time θ exists in $(0, T)$, then

$$v\,(t) = \begin{cases} 0,\ t \in [0, \theta] \\ 1,\ t \in (\theta, T]. \end{cases}$$

(The control v is bang-bang.) If not, then $v = 1$ on $[0, T]$.

7. $-b_1 + b_2 > 0$, $c_1 - b_3 < 0$. We can choose $\varepsilon > 0$ such that $-b_1 p_1 + b_2 p_2 > 0$ and $c_1 p_1 - b_3 p_3 < 0$ on $(T - \varepsilon, T]$. Then, $u = 1$ and $v = 0$ on $(T - \varepsilon, T]$. Denote

$$\tau = \inf \{t \in [0, T], \ (-b_1 p_1 + b_2 p_2)(s) > 0, (\forall)\, s \in (t, T]\}. \tag{3.14}$$

Obviously, $u(t) = 1$ on $(\tau, T]$ and

$$(-b_1 p_1 + b_2 p_2)(\tau) = 0. \tag{3.15}$$

Remark that (2.3) implies:

$$(-b_1 p_1 + b_2 p_2)' = a_1 b_1 p_1 + a_2 b_2 p_2 +$$

$$+ u(b_1 y_2 - b_2 y_1)(-b_1 p_1 + b_2 p_2) + v b_1 y_3 (c_1 p_1 - b_3 p_3), \ (\forall)\, t \in [0, T]. \tag{3.16}$$

Now, by (3.1), (3.2) and (3.15), we infer that $-b_1 p_1 + b_2 p_2$ is a monotonically increasing function in a neighborhood of τ. So, if $t < \tau$, t close to τ, we have $(-b_1 p_1 + b_2 p_2)(t) < 0$. To study the form of u and v on $[0, \tau]$, it is enough to observe that one of the following situations holds:

- if $(c_1 p_1 - b_3 p_3)(\tau) < 0$, then case 4 can be applied (for τ instead of T), to deduce that $u(t) = v(t) = 0$ on $[0, \tau]$.

- if $(c_1 p_1 - b_3 p_3)(\tau) = 0$, then case 5 (for τ instead of T) implies that $u(t) = v(t) = 0$ on $[0, \tau]$.

- if $(c_1 p_1 - b_3 p_3)(\tau) > 0$, then we can invoke case 6 to conclude that $u = 0$ on $[0, \tau]$ and v has at most one switching point θ_0 in $[0, \tau)$, i.e. either $v = 1$ on $[0, \tau]$ or

$$v(t) = \begin{cases} 0, \ t \in [0, \theta_0] \\ 1, \ t \in (\theta_0, \tau]. \end{cases}$$

Therefore, u has at most one switching point $\tau \in [0, T)$ given by (3.15). If $\tau = 0$, then $u = 1$ on $[0, T]$. If $\tau > 0$, then u is bang-bang, namely

$$u(t) = \begin{cases} 0, \ t \in [0, \tau] \\ 1, \ t \in (\tau, T]. \end{cases} \tag{3.17}$$

We now show that v has at most a finite number of switching points in the subinterval $[\tau, T]$. Indeed, denoting $z = -b_1 p_1 + b_2 p_2$ and $w = c_1 p_1 - b_3 p_3$, one derives that (z, w, p_1) is a solution of the linear differential system with continuous coefficients in each interval where u and v are constant:

$$\begin{cases} z' = [a_2 + u(b_1 y_2 - b_2 y_1)]\, z + v b_1 y_3 w + (a_1 + a_2)\, b_1 p_1 \\ w' = -c_1 y_2 u z + [-a_3 + v(b_3 y_1 - c_1 y_3)]\, w + (a_3 - a_1)\, c_1 p_1 \\ p_1' = -y_2 u z - y_3 v w - a_1 p_1. \end{cases}$$

It follows that no component of a nontrivial solution can vanish in a convergent set of points $\{t_i\}$, and that the distance between two consecutive values t_i where z, w or

p_1 vanishes is strictly positive. Consequently, $c_1p_1 - b_3p_3 < 0$ on the whole $[\tau, T]$ or $c_1p_1 - b_3p_3$ has a finite number of zeros in the subinterval $[\tau, T]$. In the first situation $v = 0$ on $[\tau, T]$ and, using the above discussion, we have in fact $v = 0$ on $[0, T]$. If the second situation takes place, let $\theta_1 < \theta_2 < ... < \theta_n$ be the zeros of $c_1p_1 - b_3p_3$ in $[\tau, T]$. Then, v is bang-bang and has one of the following forms:

$$v(t) = \begin{cases} 0, \ t \in [\theta_n, T] \\ 1, \ t \in (\theta_{n-1}, \theta_n) \\ \dots\dots\dots\dots \\ 1, \ t \in (\theta_1, \theta_2) \\ 0, \ t \in [0, \theta_1] \end{cases} \tag{3.18}$$

(if $c_1p_1 - b_3p_3 < 0$ on $(\tau, \theta_1]$ and $(c_1p_1 - b_3p_3)(\tau) \leq 0$) or

$$v(t) = \begin{cases} 0, \ t \in [\theta_n, T] \\ 1, \ t \in (\theta_{n-1}, \theta_n) \\ \dots\dots\dots\dots \\ 0, \ t \in [\theta_1, \theta_2] \\ 1, \ t \in (\theta_0, \theta_1) \\ 0, \ t \in [0, \theta_0] \end{cases} \tag{3.19}$$

(if $c_1p_1 - b_3p_3 > 0$ on $[\tau, \theta_1)$ and $0 < \theta_0 < \tau$) or

$$v(t) = \begin{cases} 0, \ t \in [\theta_n, T] \\ 1, \ t \in (\theta_{n-1}, \theta_n) \\ \dots\dots\dots\dots \\ 0, \ t \in [\theta_1, \theta_2] \\ 1, \ t \in [0, \theta_1) \end{cases} \tag{3.20}$$

(if $c_1p_1 - b_3p_3 > 0$ on $[\tau, \theta_1)$ and $\theta_0 = 0$).

8. $-b_1+b_2 > 0, c_1-b_3 > 0$. In a left neighborhood $(T-\varepsilon, T]$ of T, $-b_1p_1+b_2p_2 > 0$ and $c_1p_1 - b_3p_3 > 0$. Then, $u = v = 1$ on $(T - \varepsilon, T]$. Arguing as in the previous case, we see that u has at most one switching point $\tau \in (0, T)$, while v has at most a finite number of switching points $\theta_0 < \theta_1 < ... < \theta_n$ in $[0, T)$, where θ_0 belongs to $[0, \tau)$. Here τ is defined like in (3.14) and it verifies equation (3.15) . Moreover, if $\tau = 0$, then $u = 1$ on $[0, T]$ and if $\tau \in (0, T)$, then u has the form in (3.17) . The function v has at most one switching time θ_0 in $[0, \tau)$ and it is given by

$$v(t) = \begin{cases} 1, \ t \in [\theta_n, T] \\ 0, \ t \in (\theta_{n-1}, \theta_n) \\ \dots\dots\dots\dots \\ 1, \ t \in (\theta_1, \theta_2) \\ 0, \ t \in [0, \theta_1] \end{cases} \tag{3.21}$$

(if $c_1p_1 - b_3p_3 < 0$ on $(\tau, \theta_1]$ and $(c_1p_1 - b_3p_3)(\tau) \leq 0$) or

$$v(t) = \begin{cases} 1, \ t \in [\theta_n, T] \\ 0, \ t \in (\theta_{n-1}, \theta_n) \\ \ldots\ldots\ldots\ldots \\ 0, \ t \in [\theta_1, \theta_2] \\ 1, \ t \in (\theta_0, \theta_1) \\ 0, \ t \in [0, \theta_0] \end{cases} \tag{3.22}$$

(if $c_1p_1 - b_3p_3 > 0$ on $[\tau, \theta_1)$ and $0 < \theta_0 < \tau$) or

$$v(t) = \begin{cases} 1, \ t \in [\theta_n, T] \\ 0, \ t \in (\theta_{n-1}, \theta_n) \\ \ldots\ldots\ldots\ldots \\ 0, \ t \in [\theta_1, \theta_2] \\ 1, \ t \in [0, \theta_1) \end{cases} \tag{3.23}$$

(if $c_1p_1 - b_3p_3 > 0$ on $[\tau, \theta_1)$ and $\theta_0 = 0$).

9. $-b_1 + b_2 > 0$, $c_1 = b_3$. Obviously $u(t) = 1$ at least on a left neighborhood $(T - \varepsilon_1, T]$ of T. Since

$$(c_1p_1 - b_3p_3)' = b_3(a_3p_3 - a_1p_1) - b_3y_2(-b_1p_1 + b_2p_2) + v(b_3)^2(y_1 - y_2)(p_1 - p_3)$$

on $(T - \varepsilon_1, T]$, it follows that

$$\lim_{t \to T-0} (c_1p_1 - b_3p_3)'(t) = b_3(a_3 - a_1) - b_3(-b_1 + b_2)y_2(T).$$

The form of v depends on the sign of this limit. Denote by l the real number

$$l = a_3 - a_1 - (-b_1 + b_2)y_2(T). \tag{3.24}$$

If $l > 0$, then $c_1p_1 - b_3p_3$ is monotonically increasing in a neighborhood of T, so it is negative. The problem can be reduced to case 7.

If $l < 0$, then $c_1p_1 - b_3p_3$ is monotonically decreasing and consequently it is positive in a neighborhood of T. The problem reduces to case 8.

As a consequence of the above discussion, we can state our main result.

Theorem 3.1. Let $a_3 > a_1$, a_2, b_1, b_2, b_3, c_1 be given positive constants, (u, v) be the optimal control, (y_1, y_2, y_3) be the optimal state, and p_1, p_2, p_3 be the corresponding adjoint variables. Then, we get the following cases:

I) $-b_1 + b_2 \leq 0$, $c_1 - b_3 \leq 0$. Then $u = v = 0$ on $[0, T]$ and the optimal state is

$$y_1(t) = y_1^0 e^{a_1 t}, \ y_2(t) = y_2^0 e^{-a_2 t}, \ y_3(t) = y_3^0 e^{a_3 t}, \ t \in [0, T].$$

II) $-b_1 + b_2 \le 0$, $c_1 - b_3 > 0$. In this case, $u = 0$ on $[0, T]$ and v admits at most one switching point $\theta \in (0, T)$, which can be found from the equation $(c_1 p_1 - b_3 p_3)(\theta) = 0$.

If there is no switching point for v, then $v = 1$ on $[0, T]$. If there is a unique switching time θ, then v is a bang-bang control, namely

$$v(t) = \begin{cases} 0, & t \in [0, \theta] \\ 1, & t \in (\theta, T]. \end{cases}$$

III) $-b_1 + b_2 > 0$, $c_1 - b_3 < 0$. Then u has at most one switching point $\tau \in (0, T)$ and v has at most a finite number of switching points $\theta_0 < \theta_1 < ... < \theta_n$ in $[0, T)$, with $0 \le \theta_0 < \tau$ and $\tau \le \theta_1 < ... < \theta_n < T$.

If there is no switching time for u, then $u = 1$ on $[0, T]$. Otherwise, u is bang-bang:

$$u(t) = \begin{cases} 0, & t \in [0, \tau] \\ 1, & t \in (\tau, T], \end{cases}$$

where $\tau \in (0, T)$ is the switching time of u, given by $(-b_1 p_1 + b_2 p_2)(\tau) = 0$.

If there is no switching point for v, then $v = 0$ on $[0, T]$. Otherwise, v is a bang-bang control. It has one of the forms (3.18), (3.19), (3.20), according to the sign of $c_1 p_1 - b_3 p_3$ on $[\tau, \theta_1)$ (specified above).

IV) $-b_1 + b_2 > 0$, $c_1 - b_3 > 0$. Then, u has the same form as in case III and v has at most a finite number of switching points in $(0, T)$. If there is no switching time, then $v = 1$ on $[0, T]$. If $\theta_0 < \theta_1 < ... < \theta_n$ are its switching points (i.e. v is bang-bang), then $0 \le \theta_0 < \tau$ and $\tau \le \theta_1 < ... < \theta_n < T$. The function v is given by (3.21), (3.22) or (3.23), according to the sign of $c_1 p_1 - b_3 p_3$ on $[\tau, \theta_1)$.

V) $-b_1 + b_2 > 0$, $c_1 = b_3$. Then the problem can be reduced to case III or case IV, according to the sign of $l = a_3 - a_1 - (-b_1 + b_2) y_2(T)$.

4. Conclusions

Under the above conditions, if $-b_1 + b_2 \le 0$, $c_1 - b_3 \le 0$, then the maximization of the number of individuals is obtained if the three species are completely separated from each others.

If $-b_1 + b_2 \le 0$, $c_1 - b_3 > 0$, then the total size of the three species is maximized if the carnivorous and herbivorous species are completely separated, while the plant and the herbivorous species are either not separated at all, or completely separated in the beginning (on a time interval $[0, \theta]$) and next completely not separated (on $(\theta, T]$).

If $-b_1 + b_2 > 0$, then the herbivorous and carnivorous populations should be either completely not separated on the whole time interval $[0, T]$, or separated on $[0, \tau]$ and not separated on $(\tau, T]$, where τ is the unique switching point for u.

The herbivorous population and the plant are alternatively completely separated and completely not separated, on a finite number of time subintervals. If $c_1 < b_3$, then the herbivorous species and the plant can be also completely separated on $[0, T]$ or at least in a left neighborhood of the final time T. If $c_1 > b_3$, then the herbivorous species and the plant are not separated at all, either on the whole interval $[0, T]$ or at least on a left neighborhood of T.

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