CGF inequality and another operator equation

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Abstract: In this paper, we will prove a property of another operator equation via CGF inequality.

**Keywords:** CGF inequality, Douglas theorem

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**INTRODUCTION**

In this paper, we will prove a property of an operator equation via CGF inequality under the condition of \((p + t)s \geq 2(p + t)s_0 + r\) with \(\min\{p, 1\} + t \geq r\).

**Theorem 1.** (CGF Inequality, [Y]) Let \(p > 0, t \geq 0, r \geq 0, s > s_0 > 0\), if \(A \geq B \geq 0\), then

\[
\begin{align*}
(A^{t/2}(A^{t/2}B^p A^{t/2})^{\delta_{+r}} \leq (A^{t/2}(A^{t/2}B^p A^{t/2})^{\delta_{+r}})
\end{align*}
\]

where \(\delta = \min\{(p + t)s, 2(p + t)s_0 + \min\{\min\{p, 1\} + t, r\}\}\).

**MAIN RESULT**

**Theorem 2.**

For \(p > 0, t \geq 0, r \geq 0, s > s_0 > 0\), \((p + t)s \geq 2(p + t)s_0 + r\), \(\min\{p, 1\} + t \geq r\)

with \(2(p + t)s_0 + r = \frac{1}{n}\), if \(A \geq B \geq 0\), there exists a unique solution \(X, X > 0\)

with \(||x|| \leq 1\) s.t. \(F(s_0)XF^2(s_0)XF^2(s_0)X \cdots F^2(s_0)XF(s_0) = F(s)\),

where \(F(u) = A^{t/2}(A^{t/2}B^p A^{t/2})^{\delta_{+r}}\).
Proof. By Theorem 1, we have $F^2(s_0) \geq F^{1/n}(s)$. By Douglas Theorem in [D], there exists a unique operator $T$, $\|T\| \leq 1$, $F^{1/n}(s) = F(s_0)T = T^*F(s_0)$.

Let $X = TT^*$, then $F^{1/n}(s) = F(s_0)XF(s_0)$. It means that $F(s) = (F(s_0)XF(s_0))^n$, by which we can obtain the result.

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