Mathematical Modelling and Applied Computing (MMAC).
ISSN 0973-6093 Volume 7, Number 1 (2016), pp. 1–8

http://www.ripublication.com/mmac.htm

Improved upper bound on the $L(2, 1)$-labeling of Cartesian sum of graphs[1]

Pingli Lv[2]
Xuhai College,
China University of Mining and Technology,
Xuzhou 221008, PR China.
E-mail: lvpingli80@163.com

Bing Zhang and Ziming Duan[3]
College of Science,
China University of Mining and Technology,
Xuzhou 221116, PR China.

Abstract

An $L(2, 1)$-labeling of a graph G is defined as a function f from the vertex set $V(G)$ into the nonnegative integers such that for any two vertices x, y, $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, where $d(x, y)$ is the distance between x and y in G. The $L(2, 1)$-labeling number $\lambda_{2,1}(G)$ of G is the smallest number k such that G has an $L(2, 1)$-labeling with $k = \max\{f(x)|x \in V(G)\}$. In this paper, we consider the graph formed by the Cartesian sum of two graphs and give new upper bound of the $L(2, 1)$-labeling number, which improves the bound obtained by [Z.D.Shao, D.Zhang, The $L(2, 1)$-labeling on Cartesian sum of graphs, Applied Mathematics Letters 21(2008) 843-848.] significantly.

AMS subject classification: 05C15, 05C78.
Keywords: $L(2, 1)$-labeling, Cartesian sum, frequency assignment.

1. Introduction

An $L(2, 1)$-labeling of a graph G is defined as a function f from the vertex set $V(G)$ into the nonnegative integers such that for any two vertices x, y, $|f(x) - f(y)| \geq 2$ if

[1]Sponsored by Qing Lan Project of Jiangsu Province.
[2]Corresponding author.
[3]Supported by the Fundamental Research Funds for the Central Universities under grant 2013QNA31.

$d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, where $d(x, y)$, the distance of x and y, is the length of a shortest path between x and y. A k-$L(2, 1)$-labeling is an $L(2, 1)$-labeling such that no label is greater than k. The $L(2, 1)$-labeling number $\lambda_{2,1}(G)$ of G is the smallest number k such that G has a k-$L(2, 1)$-labeling.

The $L(2, 1)$-labeling problem comes from the frequency assignment problem. The task of the frequency assignment problem is to assign a frequency to each of the given transmitters such that the interference between nearby transmitters is avoided and the span of the assignment between the largest and smallest frequencies is minimized. Hale [1] first formulated the frequency assignment problem into a graph vertex coloring problem. Later Roberts [2] proposed a variation of the problem in which there are two levels of inference, "close" and "very close", depending on the distance between the transmitters. In order to reduce the inference, any two "close" transmitters must receive different frequencies, and any two "very close" transmitters must receive frequencies by at least two apart. The transmitters are represented by the vertices of a graph. There is an edge between two vertices if the corresponding transmitters are "very close". Two vertices are defined "close" if they are of distance two in the graph. Then in 1992, Griggs and Yeh [3] formulated the $L(2, 1)$-labeling of graphs.

The $L(2, 1)$-labeling of graphs has been extensively studied in the past decade [3-5,8-18]. Most of these papers consider the values of $\lambda_{2,1}(G)$ on particular classes of graphs. However, Griggs and Yeh [3] proved an upper bound $\Delta^2 + 2\Delta$ for any simple graph with maximum degree Δ. Later, Chang and Kuo [4] improved the bound to $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$. And Gonçalves [5] decreased it to $\Delta^2 + \Delta - 2$. In general, Griggs and Yeh [3] suggested the following conjecture:

Conjecture 1.1. For any graph G with maximum degree $\Delta \geq 2$, $\lambda_{2,1}(G) \leq \Delta^2$.

We can claim that this is the most famous open problem in this area [6]. If G is a diameter 2 graph, then $\lambda_{2,1}(G) \leq \Delta^2$ [3]. Papers [8] and [9] proved that the conjecture is true for three classes of graphs that are the direct and strong product and the cartesian product of general non trivial graphs. Van den Heuvel and McGuinnes [10] proved that the conjecture holds for planar graphs with maximum degree $\Delta \geq 7$. Sakai [11] proved that chordal graphs satisfy the conjecture and more precisely that $\lambda_{2,1}(G) \leq \frac{1}{4}(\Delta + 3)^2$. See [6, 7] for more information.

Graph products play an important role in the graph labeling problems. The Cartesian sum of two graphs G and H is the graph $G \bigoplus H$ with vertex set $V(G) \times V(H)$, and edge set

$$E(G \bigoplus H) = \{((u_1, v_1), (u_2, v_2))|u_1u_2 \in E(G) \text{ or } v_1v_2 \in E(H)]\}.$$

In this paper, we consider the graph formed by the Cartesian sum of two graphs and improve the upper bound of the $L(2, 1)$-labeling number obtained by Shao and Zhang [12] significantly.

2. A labeling algorithm

For any fixed positive integer k, a *k-stable set* of a graph G is a subset S of $V(G)$ such that every two distinct vertices in S are of distance greater than k. A 1-stable set is an usual independent set. A *maximal* 2-stable subset S of a set F is a 2-stable subset of F such that S is not a proper subset of any 2-stable subset of F.

Chang and Kuo [4] proposed the following algorithm for obtaining an $L(2,1)$-labeling and the maximum value of that labeling on a given graph.

Algorithm 2.1.
Input: A graph $G = (V, E)$.
Output: The value l is the maximum label.
Idea: In each step, find a maximal 2-stable set from these unlabeled vertices that are distance at least 2 away from those vertices labeled in the previous step. Then label all vertices in that 2-stable set with the index i in current stage. The index i starts from 0 and the increases by 1 in each step. The maximum label l is the final value of i.
Initialization: Set $S_{-1} = \emptyset$; $V = V(G)$; $i = 0$.
Iteration:
1. Determine F_i and S_i.
   $F_i = \{x \in V : x \text{ is unlabeled and } d(x,y) \geq 2 \text{ for all } y \in S_{i-1}\}$
   S_i is a maximal 2-stable subset of F_i.
   If $F_i = \emptyset$ then set $S_i = \emptyset$.
2. Label these vertices in S_i(if there is any) by i.
3. $V \leftarrow V \setminus S_i$.
4. $V \neq \emptyset$ then $i \leftarrow i + 1$; go to Step 1.
5. Record the current i as l (which is the maximal label). Stop.

Therefore l is an upper bound on $\lambda_{2,1}(G)$. We would like to find a bound in terms of the maximum degree $\Delta(G)$ of G.

Let x be a vertex with the largest label l obtained by Algorithm 2.1. Define
$I_1 = \{i : 0 \leq i \leq l-1 \text{ and } d(x,y) = 1 \text{ for some } y \in S_i\}$,
$I_2 = \{i : 0 \leq i \leq l-1 \text{ and } d(x,y) = 2 \text{ for some } y \in S_i\}$,
$\overline{I_2} = \{i : 0 \leq i \leq l-1 \text{ and } d(x,y) \leq 2 \text{ for some } y \in S_i\}$,
$I_3 = \{i : 0 \leq i \leq l-1 \text{ and } d(x,y) \geq 3 \text{ for all } y \in S_i\}$.

It is clear that $|\overline{I_2}| + |I_3| = l$. For any $i \in I_3$, $x \notin F_i$, for otherwise $S_i \cup \{x\}$ is a 2-stable of F_i, which contradicts the choice of S_i. That is, $d(x,y) = 1$ for some vertices y in S_{i-1}, i.e., $i-1 \in I_1$. This implies $|I_3| \leq |I_1|$. Hence

$$l = |\overline{I_2}| + |I_3| \leq |\overline{I_2}| + |I_1| \leq 2|I_1| + |I_2|.$$

In order to find l, it suffices to estimate $2|I_1| + |I_2|$ in terms of $\Delta(G)$.

3. The Cartesian sum of graphs

By the definition of the Cartesian sum $G \bigoplus H$ of two graphs G and H, if $\Delta(G) = 0$ or $\Delta(H) = 0$, then $G \bigoplus H$ is disjoint copies of H or G. Therefore we assume $\Delta(G) \geq 1$ and $\Delta(H) \geq 1$.

In this section, we obtain an upper bound in terms of the maximum degree of $G \bigoplus H$ for any two graphs G and H.

Theorem 3.1. Let Δ, Δ_1, Δ_2 be the maximum degree of $G \bigoplus H$, G, H and v_1, v_2 be the number of vertices of G, H respectively. Then

$$\lambda_{2,1}(G \bigoplus H) \leq \Delta^2 + v_2\Delta_1 + v_1\Delta_2 + \Delta_1\Delta_2 - \Delta_1\Delta_2^2(v_1 + 2) - \Delta_1^2\Delta_2(v_2 + 2) \\ - \Delta_1^2\Delta_2^2(v_1 + v_2 - 4).$$

Proof. Let $x = (u, v)$ be a vertex of $G \bigoplus H$ with the largest label l obtained by Algorithm 2.1. Denote

$$d = \deg_{G \bigoplus H}(x), d_1 = \deg_G(u), d_2 = \deg_H(v).$$

Then

$$d = v_2 d_1 + v_1 d_2 - d_1 d_2$$

and

$$\Delta = \Delta(G \bigoplus H) = v_2\Delta_1 + v_1\Delta_2 - \Delta_1\Delta_2.$$

The goal of the proof is to evaluate $|I_1|$ and $|I_2|$. Obviously, $|I_1| = d \leq \Delta$, $|I_2| \leq d(\Delta - 1)$. Next, we will investigate the better upper bound of $|I_2|$ by the structure of the Cartesian sum $G \bigoplus H$.

Let the number of vertices in G with distance 2 from u be r, then $0 \leq r \leq d_1(\Delta_1 - 1)$. If $r > 0$, then for any vertex u'' in G with distance 2 from u, there must be a path $uu'u''$ of length 2 between u'' and u in G. Note that $\deg_H(v) = d_2$, i.e., v has d_2 adjacent vertices in H, and so by the definition of the Cartesian sum $G \bigoplus H$, there must be $d_2 v_1 + 1$ internally disjoint paths of length 2 between (u'', v) and (u, v) in $G \bigoplus H$. Hence for any vertex in G with distance 2 from u, there must be corresponding $d_2 v_1 + 1$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \bigoplus H$. And they can only be counted once, we have to deduct $d_2 v_1$ from the value $d(\Delta - 1)$. If $r = 0$, there do not exist such corresponding $d_2 v_1 + 1$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \bigoplus H$, we have to deduct $d_2 v_1 + 1$ from the value $d(\Delta - 1)$. The maximum number of vertices with distance 2 from $x = (u, v)$ in $G \bigoplus H$ occurs when $r = d_1(\Delta_1 - 1)$, and hence in this sense the number of vertices with distance 2 from $x = (u, v)$ in $G \bigoplus H$ will decrease $d_1(\Delta_1 - 1)d_2 v_1$ from the value $d(\Delta - 1)$.

For H, we can analyze similarly with above paragraph and know that the number of vertices with distance 2 from $x = (u, v)$ in $G \bigoplus H$ will still decrease $d_2(\Delta_2 - 1)d_1 v_2$ from the value $d(\Delta - 1)$.

Also in this sense, for any vertex u'' in G with distance 2 from u and any vertex v'' in H with distance 2 from v, (u'', v'') and (u, v) are distant 2 in $G \bigoplus H$. By the definition of the Cartesian sum $G \bigoplus H$, there must be $v_1 + v_2 - 1$ internally disjoint paths of length 2 between (u'', v'') and (u, v) in $G \bigoplus H$, and be corresponding $v_1 + v_2 - 1$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \bigoplus H$. Hence the number of vertices with distance 2 from $x = (u, v)$ in $G \bigoplus H$ will still decrease $d_1(\Delta_1 - 1)d_2(\Delta_2 - 1)(v_1 + v_2 - 2)$ from the value $d(\Delta - 1)$.

Let F be the subgraph induced by the neighbors of x. Whenever there is an edge in F, the number of vertices with distance 2 from x in $G \bigoplus H$ will decrease by 2. Denote e, the number of edges of F. Next we will investigate the value of e.

For any adjacent vertex u' of u in G and any adjacent vertex v' of v in H, (u', v) and (u_s, v') (where u_s is any vertex of G) are all adjacent to (u, v) in $G \bigoplus H$. By the definition of $G \bigoplus H$, there also must be an edge between (u', v) and (u_s, v'). And there are totally d_1 neighbors (u', v) of $x = (u, v)$ and totally $v_1 d_2$ neighbors (u_s, v') of $x = (u, v)$, hence the number of edges of the subgraph F induced by the neighbors of x is at least $v_1 d_2 d_1$. By a symmetric analysis and excluding the coincided edges between (u, v') and (u', v), the edges of F should again add at least $v_2 d_1 d_2 - d_1 d_2$.

For any vertex u'' in G with distance 2 from u, there must be a path $uu'u''$ of length 2 between u'' and u in G. Then (u'', v') (where v' is adjacent to v in H) and (u', v_t) (where v_t is any vertex of H except for v) are all adjacent to (u, v) in $G \bigoplus H$. And there also must be an edge between (u'', v') and (u', v_t). Note that there are totally $d_2 d_1(\Delta_1 - 1)$ vertices (u'', v') and totally $v_2 - 1$ vertices (u', v_t) (where v_t is any vertex of H except v), hence the edges of F should again add at least $d_2 d_1(\Delta_1 - 1)(v_2 - 1)$. By a symmetric analysis and excluding the coincided edges between (u'', v') and (u', v''), the edges of F should again add at least

$$d_1 d_2(\Delta_2 - 1)(v_1 - 1) - d_1(\Delta_1 - 1)d_2(\Delta_2 - 1).$$

Therefore, we have

$$\begin{aligned} e &\geq v_1 d_2 d_1 + v_2 d_1 d_2 - d_1 d_2 + d_2 d_1(\Delta_1 - 1)(v_2 - 1) \\ &\quad + d_1 d_2(\Delta_2 - 1)(v_1 - 1) - d_1(\Delta_1 - 1)d_2(\Delta_2 - 1) \\ &= d_1 d_2(\Delta_2 v_1 + \Delta_1 v_2 - \Delta_1 \Delta_2). \end{aligned}$$

Hence the number of vertices with distance 2 from $x = (u, v)$ in $G \bigoplus H$ will decrease

$$\begin{aligned}&d_1(\Delta_1-1)d_2v_1+d_2(\Delta_2-1)d_1v_2\\&\quad+d_1(\Delta_1-1)d_2(\Delta_2-1)(v_1+v_2-2)\\&\quad+2d_1d_2(\Delta_2v_1+\Delta_1v_2-\Delta_1\Delta_2)\end{aligned}$$

from the value $d(\Delta-1)$ altogether.

Thus, we have $|I_1|=d\le\Delta$,

$$\begin{aligned}|I_2|\le\ &d(\Delta-1)-d_1(\Delta_1-1)d_2v_1-d_2(\Delta_2-1)d_1v_2\\&-d_1(\Delta_1-1)d_2(\Delta_2-1)(v_1+v_2-2)\\&-2d_1d_2(\Delta_2v_1+\Delta_1v_2-\Delta_1\Delta_2).\end{aligned}$$

Then

$$\begin{aligned}l\le 2|I_1|+|I_2|\le\ &2d+d(\Delta-1)-d_1(\Delta_1-1)d_2v_1\\&-d_2(\Delta_2-1)d_1v_2-d_1(\Delta_1-1)d_2(\Delta_2-1)(v_1+v_2-2)\\&-2d_1d_2(\Delta_2v_1+\Delta_1v_2-\Delta_1\Delta_2)\\&\quad=(v_2d_1+v_1d_2-d_1d_2)(\Delta+1)-d_1(\Delta_1-1)d_2v_1\\&-d_2(\Delta_2-1)d_1v_2-d_1(\Delta_1-1)d_2(\Delta_2-1)(v_1+v_2-2)\\&-2d_1d_2(\Delta_2v_1+\Delta_1v_2-\Delta_1\Delta_2)\triangleq f(d_1,d_2).\end{aligned}$$

Then $f(d_1,d_2)$ has the absolute maximum when $d_1=\Delta_1, d_2=\Delta_2$ for $0\le d_1\le\Delta_1, 0\le d_2\le\Delta_2$. And

$$\begin{aligned}f(\Delta_1,\Delta_2)\ &=\ \Delta^2+v_2\Delta_1+v_1\Delta_2-\Delta_1\Delta_2-\Delta_1(\Delta_1-1)\Delta_2v_1\\&\quad-\Delta_2(\Delta_2-1)\Delta_1v_2-\Delta_1(\Delta_1-1)\Delta_2(\Delta_2-1)(v_1+v_2-2)\\&\quad-2\Delta_1\Delta_2(\Delta_2v_1+\Delta_1v_2-\Delta_1\Delta_2)\\&=\ \Delta^2+v_2\Delta_1+v_1\Delta_2-\Delta_1\Delta_2(\Delta_2v_1+\Delta_1v_2-4\Delta_1\Delta_2\\&\quad+\Delta_1\Delta_2v_1+\Delta_1\Delta_2v_2+2\Delta_1+2\Delta_2-1)\\&=\ \Delta^2+v_2\Delta_1+v_1\Delta_2+\Delta_1\Delta_2-\Delta_1\Delta_2^2(v_1+2)\\&\quad-\Delta_1^2\Delta_2(v_2+2)-\Delta_1^2\Delta_2^2(v_1+v_2-4).\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_{2,1}(G\bigoplus H)\le l\le\ &\Delta^2+v_2\Delta_1+v_1\Delta_2+\Delta_1\Delta_2\\&-\Delta_1\Delta_2^2(v_1+2)-\Delta_1^2\Delta_2(v_2+2)\\&-\Delta_1^2\Delta_2^2(v_1+v_2-4).\end{aligned}$$

In [12], it is proved that

$$\begin{aligned}\lambda_{2,1}(G\bigoplus H)\le\ &\Delta^2-v_1(\Delta_1-1)\Delta_2-v_2(\Delta_2-1)\Delta_1\\&-(\Delta_1+\Delta_2)\Delta_1\Delta_2-\Delta_1-\Delta_2+1.\end{aligned}$$

Because

$$\begin{aligned}
\Delta^2 &- v_1(\Delta_1 - 1)\Delta_2 - v_2(\Delta_2 - 1)\Delta_1 \\
&- (\Delta_1 + \Delta_2)\Delta_1\Delta_2 - \Delta_1 - \Delta_2 + 1 \\
&- (\Delta^2 + v_2\Delta_1 + v_1\Delta_2 + \Delta_1\Delta_2 \\
&- \Delta_1\Delta_2^2(v_1 + 2) - \Delta_1^2\Delta_2(v_2 + 2) \\
&- \Delta_1^2\Delta_2^2(v_1 + v_2 - 4)) \\
&= (\Delta_2 - 1)(\Delta_1(\Delta_2 + 1) - 1) \\
&+ \Delta_1\Delta_2(\Delta_1 + \Delta_2 v_1 + \Delta_1 v_2 - 5) \\
&+ \Delta_1\Delta_2(\Delta_1\Delta_2 - 1)(v_1 + v_2 - 4) \geq 0,
\end{aligned}$$

we reduce the bound significantly. ■

References

[1] W.K. Hale, Frequency assignment: theory and application, Proc. IEEE, 68 (1980), 1497–1514.

[2] F.S. Roberts, T-coloring of graphs: recent results and open problem, Discrete Math., 93 (1991), 229–245.

[3] J.R. Griggs, R.K. Yeh, Labeling graphs with a condition at distance two, SIAM J. Discrete Math., 5 (1992), 586–595.

[4] G.J. Chang, D. Kuo, The $L(2,1)$-labeling on graphs, SIAM J. Discrete Math., 9 (1996) 309–316.

[5] D. Gonçalves, On the $L(p,1)$-labeling of graphs, Discrete Math., 308 (2008), 1405–1414.

[6] T. Calamoneri, The $L(h,k)$-labeling problem: an survey and annotated bibliography, The Computer Journal, 49 (2006), 585–608.

[7] R.K. Yeh, A survey on labeling graphs with a condition at distance two, Discrete Math., 306 (2006), 1217–1231.

[8] S.Klavžar, S.Špacapan, The Δ^2-conjecture for $L(2,1)$-labelings is true for directed and strong products of graphs, IEEE Trans. Circuits and Systems II, 53 (2006), 274–277.

[9] Z.Shao, R.K.Yeh, The $L(2,1)$-labeling and operations of graphs, IEEE Trans. Circuits Syst. I Fund. Theory Appl., 52 (2005), 668–671.

[10] J. van den Heuvel, S. McGuinnes, Colouring the square of a planar graphs, J. of Graph Theory, 42 (2003), 110–124.

[11] D. Sakai, Labeling Chordal Graphs: Distance Two Condition, SIAM J. Disc. Math., 7 (1994), 133–140.

[12] Z.D. Shao, D. Zhang, The $L(2, 1)$-labeling on Cartesian sum of graphs, Appl. Math. Lett., 21 (2008), 843–848.

[13] J.P. Georges, D.W. Mauro, Generalized vertex labelings with a condition at distance two, Congr. Numer., 109 (1995), 141–159.

[14] J. Heuvel, R.A. Leese, M.A. Shepherd, Graph labeling and radio channel assignment, J. Graph Theory, 29 (1998), 263–283.

[15] G.J. Chang, S.C. Liaw, The $L(2, 1)$-labeling problem on ditrees, Ars Combin., 66 (2003), 23–31.

[16] W.F. Wang, The $L(2, 1)$-labelling of trees, Discrete Appl. Math., 154 (2006), 598–603.

[17] T. Hasunuma, T. Ishii, H. Ono, Y. Uno, An $o(n^{1.75})$ algorithm for $L(2, 1)$-labeling of trees, Theoretical Computer Science, 410 (2009), 3702–3710.

[18] Z.D. Shao, R.K. Yeh, K.K. Poon, W.C. Shiu, The $L(2, 1)$-labeling of $K_{1,n}$-free graphs and its applications, Appl. Math. Lett., 21 (2008), 1188–1193.