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The $\mathrm{H}^{\mathrm{p}}(\mathbb{R})$ −Boundedness of Hausdorff Operator Involving Wavelet Transformation

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Abstract

H^p ($\mathbb{R}$)-boundedness of Hausdorff operator is obtained by using the method of atomic decomposition, molecular characterization and wavelet transform for $0 < p \leq 1$.

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Introduction

E.Liflyand and F.Moricz [1] investigated the boundedness of Hausdorff operator $H_\varphi f$ of a function $f \in L^1(\mathbb{R})$ generated by $\varphi \in L^1(\mathbb{R})$ on Hardy space of type $H^1(\mathbb{R})$ by using the theory of Hilbert transformation and Fourier Transformation.

Kanjin [2] obtained the $H^p(\mathbb{R})$-boundedness of the Hausdorff operator and Cesaro operator by using the method of atomic decomposition, molecular characterization and Fourier transform for $0< p \leq 1$.

From [2] the real Hardy space $H^p(\mathbb{R})$ is defined to be the space of boundary distributions *f(x)=Re(F(z))* of the real parts of function *F(z)* as *z = x+it* in the Hardy space $H^p(\mathbb{R}^2_+)$={F(z):F is analytic in $\mathbb{R}^2_+$} and $\|F\|_{H^p(\mathbb{R}^2_+)}$=$\sup_{t>0}(\int_{-\infty}^{+\infty}|F(x+it|^p\,dx)^{1/p}$ for $0 < p \leq 1$ on the upper half plane $\mathbb{R}^2_+$ with the norm

$$\|f\|_{H^p(\mathbb{R})} = \|F\|_{H^p(\mathbb{R}^2_+)} \tag{1.1}$$

Now we recall the definition of Hausdorff operator from Y.Kanjin [2].

Let $f, \varphi \in L^1(\mathbb{R})$. Then, the Hausdorff operator H_φ of a function f in $\mathbb{R}$ generated by φ is defined in such a way that its Fourier transform satisfies the following:

$$(H_\varphi f)^\wedge(t)=\int_{-\infty}^{\infty} \hat{f}\,(t\xi)\varphi(\xi)d\xi, t \in \mathbb{R}. \tag{1.2}$$

For α =1, 2, 3… the Cesaro operator C_α of order α is given by

$$\widehat{C_\alpha}f(t) = \begin{cases} \frac{\alpha}{t^\alpha} \int_0^t \hat{f}(\xi)\,(t-\xi)^{\alpha-1}\, d\xi, (t \neq 0) \\ \hat{f}(0)\ (t=0) \end{cases},$$

Here $C_\alpha = H_{\varphi_\alpha}$ when $\varphi_\alpha(\xi) = \alpha(1-\xi)^{\alpha-1}\ \chi_{(0,1)}(\xi)$ where $\chi_{(0,1)}$ is the characteristic function of the interval $(0,1)$.

Now we recall the definition of continuous wavelet transform from [3, pp-366-367].

If $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\psi_{a,b}(t)$ is defined by the following way:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) \text{ where } a \in \mathbb{R}^+, b \in \mathbb{R}. \tag{1.3}$$

Then the wavelet transform W_ψ of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by $(W_\psi f)(b,a) = \langle f, \psi_{a,b}\rangle = \int_{-\infty}^{\infty} f(t)\overline{\psi_{a,b}}(t)\,dt$ for $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$.

Using the Parseval formula for the Fourier transform we have

$$\begin{aligned}(W_\psi f)(b,a) &= \langle f, \psi_{a,b}\rangle = \langle \hat{f}, \hat{\psi}_{a,b}\rangle \\ &= \sqrt{|a|}\int_{-\infty}^{\infty} \exp(ibw)\,\hat{f}(w)\bar{\hat{\psi}}(aw)dw.\end{aligned} \tag{1.4}$$

Let $\varphi \in L^1(\mathbb{R})$. Then the wavelet transform of a function $H_\varphi f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$W_\psi(H_\varphi f)(b,a) = \int_{-\infty}^{\infty} \exp(ibw)(H_\varphi\, f)^\wedge(w)\bar{\hat{\psi}}(aw)dw. \tag{1.5}$$

Now we give the definition of atomic decomposition and molecular characterization of $H^p(R)$ from [2, p p.38-39]

Definition1.1 Let $0 < p \leq 1$and k be an integer such that $k \geq \frac{1}{p} - 1$. A real valued function $a^0(x)$ is called a $(p,2,k)$ atom if

(1) $a^0(x)$ is supported in an interval [c, c+h]

(2) $\|a^0\|_2 = (\int_{-\infty}^{\infty} |a^0(x)|^2\,dx)^{\frac{1}{2}} \leq h^{\frac{p-2}{2p}}$

(3) $\int_{-\infty}^{\infty} x^j a^0(x)dx = 0$ for $j = 0,1,2,3,\ldots.k$.Then the atomic decomposition says that for $f \in H^p(\mathbb{R})$ there exist a sequence (a_j^0) of $(p,2,k)$ atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j |\lambda_j|^p \leq C\|f\|_{H^p(\mathbb{R})}^p$such that $f = \sum_j \lambda_j\, a_j^0$ the series converging to $f \in H^p(R)$ and also in the sense of tempered distributions.

A real valued function $M(x)$ is called a $(p,2,b)$ molecule centred at x_0 if
(1) $N(M) = \|M\|_2^{1-\theta}\|M(.-x_0)^b\|_2^{\theta} < \infty$
where $b > \frac{2-p}{2p}, \theta = \frac{2-p}{2pb}$,

(2) $\int_{-\infty}^{\infty} x^j\, M(x)dx = 0, j = 0,1,2,3, \ldots.. \left[\frac{1}{p} - 1\right]$.
$N(M)$ is called the molecular norm of $M(x)$.

The molecular characterization asserts that if $f = \sum_j M_j$ with $(p,2,b)$ molecules M_j as tempered distributions and $\sum_j N(M_j)^p < \infty$, then $f \in H^p(\mathbb{R})$ and $\|f\|_{H^p(\mathbb{R})}^p \leq C \sum_j N(M_j)^p$.

Hausdorff Operator

In this section we study the boundedness property of Hausdorff operator on Hardy space $H^p(R)$ by using the theory of wavelet transformation

Theorem 2.1 Let $0 < p \leq 1$ and r be the smallest integer such that $r > \frac{2-p}{2p}$.Suppose that $\varphi \in L^1(R)$ satisfies the following conditions

$$\int_{-\infty}^{\infty} (|\xi|)^{-1/2}|\phi(\xi)|d\,\xi < \infty,$$

$\varphi \in C^{2r}(R)$ with $sup_x|x|^r|\hat{\varphi}^r(x)| < \infty, sup_x|x|^{2r}|\hat{\varphi}^{2r}(x)| < \infty$.

Let ψ be a wavelet of which derivatives of Fourier transform satisfy the following condition

$$|D_w^r\,\hat{\psi}(aw)| \leq C_r(1 + aw)^{-l}, l > 0.$$

Then for $a^0(p,2,r-1)$-atom $a^0, W_\psi H_\varphi a^0$ is $a^0(p,2,r)$-molecule centered at 0 and $N(W_\psi H_\varphi a^0) \leq C$, where C is independent of the atoms a^0.

Proof: Let $0 < p \leq 1$ and r be the smallest integer such that $r > \frac{2-p}{2p}$ then we begin with estimating the molecular norm $N(W_\psi H_\varphi f)$ for a $(p,2,r-1)$ atom of a function f, Using (1.5) we get

$$(ib)^r(W_\psi H_\varphi a^0)(b,a) = \int_{-\infty}^{\infty} D_w^r(e^{ibw})\hat{\psi}(aw)\,(\widehat{H_\varphi a^0})(aw)\,dw.$$

By integration by parts we have

$$(ib)^r(W_\psi H_\varphi a^0)(b,a) = (-1)^r \int_{-\infty}^{\infty}(e^{ibw})D_w^r[\,\hat{\psi}(aw)(\widehat{H_\varphi a^0})(aw)]\,dw$$
$$= (-1)^r F[D_w^r[\hat{\psi}(aw)(\widehat{H_\varphi a^0})(aw)]]\,(b).$$

Therefore,

$$\left\|(ib)^r\left(W_\psi H_\varphi a^0\right)(b,a)\right\|_2 = \left\|F[D_w^r\hat{\psi}(aw)(H_\varphi a^0\hat{)}(aw)(b)]\right\|_2.$$

By Plancherel formula, the right hand side gives

$$\left\|(ib)^r\left(W_\psi H_\varphi a^0\right)(b,a)\right\|_2 = \|D_w^r[\hat{\psi}(aw)(H_\varphi a^0\hat{)}(aw)\|]_2$$

$$= \left\|\sum_{n=0}^{r}\binom{r}{n} D_w^{r-n}[(H_\varphi a^0)^\wedge(w)]D_w^n\,\hat{\psi}(aw)\right\|_2$$

$$=(\int_{-\infty}^{\infty}\left|\sum_{n=0}^{r}\binom{r}{n} D_w^{r-n}[(H_\varphi a^0)^\wedge(w)]D_w^n[\hat{\psi}(aw)]\right|^2 dw)^{\frac{1}{2}}.$$

$$\leq (\int_{-\infty}^{\infty}\left|\sum_{n=0}^{r}\binom{r}{n} D_w^{r-n}[(H_\varphi a^0)^\wedge(w)]C_n(1+|aw|)^{-l}]\right|^2 dw)^{\frac{1}{2}}$$

$$\leq C_n sup_w(1+|aw|)^{-l}(\int_{-\infty}^{\infty}\left|\sum_{n=0}^{r}\binom{r}{n}\left(\frac{d}{dw}\right)^{r-n}[(H_\varphi a^0)^\wedge(w)]]\right|^2 dw)^{\frac{1}{2}}.$$

Applying [2, Lemma2.1pp.40-41] we have

$$\left\|(ib)^r\left(W_\psi H_\varphi a^0\right)(b,a)\right\|_2 \leq C_{n,l}\|a^0\|_2^\delta. \tag{2.1}$$

$$= L < \infty.$$

Now, by (1.5) we have

$$\left\|\left(W_\psi H_\varphi a^0\right)(b,a)\right\|_2 = \left\|F[\hat{\psi}(aw)(H_\varphi a^0)^\wedge(w)](b)\right\|_2$$

$$=\left\|\hat{\psi}(aw)(H_\varphi a^0)^\wedge(w)\right\|_2$$

$$\leq \left\|C_0(1+|aw|)^{-l}(H_\varphi a^0)^\wedge(w)\right\|_2$$

$$\leq C_0 sup_w(1+|aw|)^{-l}\left\|(H_\varphi a^0)^\wedge(w)\right\|_2$$

$$\leq C_0\left\|(H_\varphi a^0)^\wedge(w)\right\|_2.$$

From [2, p.40] we have

$$\left\|\left(W_\psi H_\varphi a^0\right)(b,a)\right\|_2 \leq C_0 D_0\|a^0\|_2,$$

$$=M < \infty$$

where $D_0 = \int_{-\infty}^{\infty}(|\xi|)^{-1/2}|\varphi(\xi)|d\,\xi$, which yields

$$N\left(W_\psi H_\varphi f\right) \leq (M)^{1-\theta}\left\|D_w^r[\hat{\psi}(aw)(H_\varphi a^0\hat{)}(aw)]\right\|_2^{\theta}.$$

From (2.1) and using the definition 1.1 we can find

$$N\left(W_\psi H_\varphi f\right) \leq M^{1-\theta}L^\theta \leq C$$

where *C* is a constant and independent of function f.

Theorem 2.2: Let $0 < p \leq 1, \varphi \in L^1(\mathbb{R})$ and $f = \sum_{j=0}^{\infty}\lambda_j\, a_j^0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfies the same conditions as in the theorem 2.1.Then the wavelet transform $W_\psi H_\varphi f$ is bounded on $H^p(\mathbb{R})$.

Proof: We first discuss about the definition of $W_\psi H_\varphi f$ for $f \in H^p(\mathbb{R}), 0 < p \leq 1.$ We use the fact that a function of Lipschitz space $\wedge_{\frac{1}{p}-1}(\mathbb{R})$ defines a continuous linear functional on $H^p(\mathbb{R})$.

Let $0 < p \leq 1$, and $r > \frac{1}{p} - \frac{1}{2}$ be the smallest integer and suppose that φ satisfies the same conditions as in the theorem 2.1.

We put

$$\hat{\varphi}_t(x) = \hat{\varphi}(tx).$$

Then for usual calculation we have

$$\left|\left(\frac{d}{dx}\right)^j \hat{\varphi}_t(x)\right| \leq A_j\ |t|^j,$$

where

$A_j = \ sup_x\left|\hat{\varphi}^j(x)\right|$ and the constants A_j are finite.

Hence,

$$\|\hat{\varphi}_t\|_{\wedge_{\frac{1}{p}-1}(\mathbb{R})} \leq C(1+\ |t|^r).$$

Let *m(t)* = $< \hat{\varphi}_t(w), \hat{\psi}_{a,b}(w) >$. Then, by Schwartz inequality we have

$$|m(t)| \leq \|\hat{\varphi}_t\|_2\left\|\hat{\psi}_{a,b}\right\|_2.$$

By the isometry of wavelet, the above expression yields

$$|m(t)|\ \leq \|\hat{\varphi}_t\|_2\|\psi\|_2.$$

Hence, we can write

$$|m(t)|\ \leq C(1+\ |t|^r)\|\psi\|_2.$$

This implies that,

$$m(t) = \langle\hat{\varphi}_t(w), \hat{\psi}_{a,b}(w)\rangle \in\ \wedge_{\frac{1}{p}-1}(\mathbb{R}).$$

For every $t \in R$, the function $m(t)$defines a continuous linear functional on $H^p(R)$ and satisfies the following inequality

$$|\langle f(t), m(t)\rangle| \leq C(1+|t|^r)\|\psi\|_2\|f\|_{H^p(R)},$$

where <., .> is the duality paring between $H^p(\mathbb{R})$ and $\wedge_{\frac{1}{p}-1}$ $(\mathbb{R})$ and C is independent of f and t.

Now we define the wavelet transform of Hausdorff operator of a function *f* by

$$\begin{aligned}(W_\psi H_\varphi f)(b,a) &= \langle H_\varphi f, \psi_{a,b}(t)\rangle \\ &= \langle (H_\varphi f)^\wedge(w), \hat{\psi}_{a,b}(w)\rangle \\ &= \langle\langle f, \hat{\varphi}_t\rangle, \hat{\psi}_{a,b}\rangle \\ &= \langle f, \langle\hat{\varphi}_t, \hat{\psi}_{a,b}\rangle\rangle \\ &= \langle f(t), m(t)\rangle.\end{aligned}$$

Now we turn to the proof of the theorem.

Let $0 < p \leq 1$ and r be the smallest integer such that $r > \frac{1}{p} - \frac{1}{2}$.Let $f \in H^p(R)$, we have an atomic decomposition $f = \sum_{j=0}^{\infty} \lambda_j\, a_j^0$, where $\sum_{j=0}^{\infty} |\lambda_j|^p \leq C\|f\|_{H^p(\mathbb{R})}^p$ and a_j^0 is a $(p, 2, r-1)$ atom. From the theorem 2.1, we have

$$\sum_{j=0}^{\infty} N(\lambda_j W_\psi H_\varphi a_j^0)^p = \sum_{j=0}^{\infty} |\lambda_j|^p N(W_\psi H_\varphi a_j^0)^p$$
$$\leq C \sum_{j=0}^{\infty} |\lambda_j|^p$$
$$\leq C\, \|f\|_{H^p(\mathbb{R})}^p.$$

Thus, the series $\sum_{j=0}^{\infty} \lambda_j\, W_\psi H_\varphi a_j$ converges to a tempered distribution g in S' and $\|g\|_{\,H^p(\mathbb{R})} \leq C\|f\|_{H^p(\mathbb{R})}$, where S is the Schwartz space.

Now it is enough to show that $g = W_\psi H_\varphi f$ in S'. Let $h(b) \in S$, it follows that

$$< g, h >= \sum_{j=0}^{\infty} \lambda_j \langle W_\psi H_\varphi a_j^0, h\rangle$$
$$< g, h >= \sum_{j=0}^{\infty} \lambda_j \langle W_\psi H_\varphi a_j^0, h\rangle$$
$$= \sum_{j=0}^{\infty} \lambda_j \int_{-\infty}^{\infty} W_\psi\, H_\varphi a_j^0(b, a) h(b) db$$
$$= \sum_{j=0}^{\infty} \lambda_j \int_{-\infty}^{\infty} \langle a_j^0(t), m(t)\rangle\, h(b) db,$$

where $\langle , \rangle$ is the duality paring between S and S'.

By the fact that$\left|\langle a_j^0(t), m(t)\rangle\right| \leq C\|m(t)\|_{\wedge_{\frac{1}{p}-1}(\mathbb{R})}$.

We can change the order of sum and integral in the last term, it follows that

$$< g, h >= \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \lambda_j \langle a_j^0(t), m(t)\rangle h(b) db$$
$$= \int_{-\infty}^{\infty} \langle f(t), m(t)\rangle h(b) db$$
$$= \int_{-\infty}^{\infty} (W_\psi\, H_\varphi f)(b, a) h(b) db$$
$$= \langle W_\psi H_\varphi f, h\rangle.$$

Therefore,

$$g = W_\psi H_\varphi f \text{ in } S'.$$

This completes the proof of the theorem.

Theorem 2.3: The wavelet transform of Cesaro operator is bounded on $H^p(R)$ for $\frac{2}{(2\alpha+1)} < p \leq 1$.

Proof: We put $\varphi_\alpha(\xi) = \alpha(1-\xi)^{\alpha-1}\chi_{(0,1)}(\xi)$.

Then,

$$C_\alpha = H_{\varphi_\alpha} \Rightarrow W_\psi C_\alpha = W_\psi H_{\varphi_\alpha}.$$

To prove $\varphi_\alpha \in L^1(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} |\varphi_\alpha(\xi)|\, d\xi = \int_{-\infty}^{\infty} \left|\alpha(1-\xi)^{\alpha-1}\chi_{(0,1)}(\xi)\right| d\xi$$
$$= \int_0^1 \alpha(1-\xi)^{\alpha-1}\, d\xi < \infty.$$

Next, $\hat{\varphi}_\alpha(x)=\int_{-\infty}^{\infty} \exp(-ix\xi)\alpha(1-\xi)^{\alpha-1}X_{0,1}(\xi)d\xi$
$$=\int_0^1 \exp(-ix\xi)\alpha(1-\xi)^{\alpha-1}\, d\xi.$$

Now we put $1-\xi = t,$ then

$$\hat{\varphi}_\alpha(x) = \int_0^1 \exp(-ix(1-t))\alpha t^{\alpha-1}\ dt$$
$$= \int_0^1 [1 - ix(1-t) + \frac{\left(ix(1-t)\right)^2}{2} - \frac{\left(ix(1-t)\right)^3}{6} + \cdots.]\alpha t^{\alpha-1}dt$$
$$= \alpha!\,[\frac{1}{\alpha!} + \frac{-ix}{(\alpha+1)!} + \frac{(-ix)^2}{(\alpha+2)!} + \frac{(-ix)^3}{(\alpha+3)!} + \cdots..]$$
$$= \alpha!\,(-ix)^{-\alpha}[\exp(-ix) - \sum_{j=0}^{\alpha-1}\frac{(-ix)^j}{j!}].$$

Hence φ_α satisfies both the conditions of theorem 2.1. Now using the theorem 2.1, theorem 2.2 and Kanjin [2], we find that $W_\psi H_{\varphi_\alpha}$is bounded on $H^p(R)$ for $\frac{2}{(2\alpha+1)} < p \le 1$.

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References

[1] E. Liflyand, F.Moricz, The Hausdorff operator is bounded on the real Hardy Space $H^1(R)$, Proc.Amer.Math.Soc.128 (1999), 1391-1396.

[2] Y. Kanjin The Haussdorff operator on the real Hardy space $H^p(R)$, Studia Math, 148(2001), 37-45.

[3] L.Debnath, Wavelet Transform and their applications, Birkhauser Boston 2002.