Journal of Wavelet Theory and Applications.
ISSN 0973-6336 Volume 1 Number 1 (2007), pp. 13–30

http://www.ripublication.com/jwta.htm

Chebyshev Wavelet Transforms

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Abstract

Using convolution theory for the Chebyshev transform due to Butzer and Stens, continuous Chebyshev wavelet transform (CWT) is defined.

A general reconstruction formula is established. A discrete version of the CWT is also given and some of its properties are investigated.

Keywords: Chebyshev polynomial, Chebyshev transform, Chebyshev convolution, wavelet transforms, Chebyshev wavelet transform.

1. Introduction

Integral transforms involving special functions as kernels have been used by many authors for the construction of wavelets and wavelet transforms. Pathak and Dixit [8] have constructed Bessel wavelets and Bessel wavelet transforms using theory of Hankel convolution. Wavelets on finite intervals involving solution of certain Sturm-Liouville system have been studied by Depczynski [5]. Motivated from these works we wish to develop a theory of wavelets and wavelet transforms involving Chebyshev polynomials. These polynomials have been used extensively by Butzer et al [2] in approximation theory. From these papers we recall certain definitions and properties of Chebyshev polynomial and Chebyshev transforms which form the basis of this work.

Let X denote the space $L_w^p(-1,1)$, $1 \le p < \infty$, or $C[-1,1]$ endowed with the norms

$$\| f \|_p = \| f \|_{L_w^p} = \left[\frac{1}{\pi} \int_{-1}^{1} | f(x) |^p \; w(x)dx \right]^{1/p} < \infty, \qquad 1 \le p < \infty; \tag{1.1}$$

$$\| f \|_C = \sup_{-1 \le x \le 1} | f(x) |, \tag{1.2}$$

where

$$w(x)=(1-x^2)^{-1/2}.$$

An inner product on X is given by

$$<f,\ g>_w\ =\ \frac{1}{\pi}\int_{-1}^{1} f(x)\ \overline{g(x)}\, w(x)\ dx. \tag{1.3}$$

As usual we denote the Chebyshev polynomial by

$$T_n(x)=\cos(n\ \cos^{-1}x),\quad n\in \mathbf{N}_0;\quad x\in[-1,1]. \tag{1.4}$$

T_n is a polynomial of degree n, $\|T_n\|_C$ = 1 and the $\{T_n\}$ satisfy the orthogonality relation

$$\frac{1}{\pi}\int_{-1}^{1} T_n(x)\ T_m(x)\, w(x)\, dx = \begin{cases} 1 & m=n=0 \\ \frac{1}{2} & m=n\neq 0 \\ 0 & m\neq n \end{cases} \tag{1.5}$$

The Chebyshev transform of a function $f\in X$ is defined by

$$\Im[f](k)=\hat{f}(k)=\frac{1}{\pi}\int_{-1}^{1} f(x)\, T_k(x) w(x) dx,\quad k\in \mathbf{N}_0. \tag{1.6}$$

The operator $\Im$ associates to each $f\in X$ sequence of real numbers $\left\{\hat{f}(k)\right\}_{k=0}^{\infty}$, called Fourier – Chebyshev coefficients.

By putting t = cos θ in (1.6) and using the fact that

$$\int_{-\pi}^{\pi} f(\cos\theta)\ \sin\ k\,\theta\ \ d\theta\ = 0,$$

we have

$$\Im[f](k)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\cos\ \theta)\ \bar{e}^{ik\theta}\ \ d\theta\ = F[f o\ \cos](k), \tag{1.7}$$

where F[f o cos] (k) denotes the Fourier transform of the function f(cos θ).

The inverse Chebyshev transform is given by

$$\Im[f]^{v}(x)=f(x)=\hat{f}(0)+2\sum_{k=1}^{\infty}\hat{f}(k)T_k(x). \tag{1.8}$$

Lemma 1.1 Assume f, g $\in X$ and $c\in R$, then

(i) $\left|\hat{f}(k)\right| \leq \|f\|_1,\quad k\in \mathbf{N}_0;$

(ii) $(f+g)^{\wedge}(k)=\hat{f}(k)+\hat{g}(k),\ (cf)^{\wedge}(k)=c\hat{f}(k);$

(iii) $\lim\limits_{k\to\infty}\hat{f}(k)=0;$

(iv) $\hat{f}(k)=0, k\in \mathbf{N}_0$ iff $f(x)=0$ a.e.;

(v) $(T_n)^{\wedge}(k)=\begin{cases}1 & k=n=0\\ \dfrac{1}{2} & k=n\neq 0\\ 0 & k\neq n.\end{cases}$

For a proof we may refer to [1].

Let us now define the translation operator τ_y which plays an important role in our investigation. For $f\in X$ and $|y|\leq 1$, the translation operator τ_y is defined by

$$(\tau_y f)(x)=f(x,y)=\frac{1}{2}\left\{f(xy+[(1-x^2)(1-y^2)]^{1/2})+f(xy-[(1-x^2)(1-y^2)]^{1/2})\right\} \qquad (1.9)$$

The translation operator τ_y possesses the following properties.

Lemma 1.2. Let $f\in X$ and $|y|\leq 1$. Then

(i) $\|\tau_y f\|_X \leq \|f\|_X;$

(ii) $(\tau_y f)(x)=(\tau_x f)(y);$

(iii) $\lim\limits_{y\to\infty}\|\tau_y f-f\|_X=0;$

(iv) $[\tau_y f]^{\wedge}(k)=T_k(y)\ \hat{f}(k);$

(v) $(\tau_y T_n)(x)=T_n(y)T_n(x), \qquad n\in \mathbf{N}_0$

The proof can be found in [1].

As in [1], for functions f, g defined and measurable on [-1,1], the Chebyshev convolution is given by

$$(f*g)\ (y)\ =\frac{1}{\pi}\int_{-1}^{1}(\tau_y f)(x)\ g(x)\ w(x)dx. \qquad (1.10)$$

Theorem 1.3. If $f\in X$, $g\in L^1_w$, then (f*g) exists (a.e.) and belongs to X. Moreover,

$$\|f*g\|_X \leq \|f\|_X\|g\|_1; \qquad (1.11)$$

$$(f*g)^{\wedge}(k)=\hat{f}(k)\ \ \hat{g}(k). \qquad (1.12)$$

The proof can be found in [1].

For any $f\in L^2_w(-1,1)$, the following Parseval identity holds for Chebyshev transform:

$$\left|\hat{f}(0)\right|^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left|\hat{f}(k)\right|^2 = \| f \|_2^2 \, . \tag{1.13}$$

In this paper, motivated from the work on classical wavelet transforms (cf. [4], [6]) and more recent works (cf. [7], [8]), we define the Chebyshev wavelets and Chebyshev wavelet transform and study their properties.

A general reconstruction formula is derived. A semi-discrete Chebyshev wavelet transform is defined and a reconstruction formula, under a suitable stability condition, is obtained. Furthermore, discrete Chebyshev wavelet transform is investigated. Using Chebyshev wavelet, frame and Riesz basis [4] are also studied. A few examples of CWT are given.

2. Chebyshev Wavelet Transform

For a function $\psi \in X,$ define the dilation D_a by

$$D_a \psi(t) = \psi(at), \quad o < a \leq 1\,. \tag{2.1}$$

Using the translation operator (1.9) and the above dilation, the Chebyshev wavelet $\psi_{b,a}$ (t) is defined as follows

$$\psi_{b,a}(t) = \tau_b \; D_a \; \psi(t) = \tau_b \psi(at) = \psi(at, b)\,. \tag{2.2}$$

$$= \frac{1}{2}\left\{\psi(abt + \left[(1-a^2t^2)(1-b^2)\right]^{1/2}) + \psi(abt + \left[(1-a^2t^2)(1-b^2)\right]^{1/2})\right\} \tag{2.3}$$

where $-1 \leq b \leq 1$ and $0 < a \leq 1.$

Now, using the wavelet $\psi_{b,a}$ the Chebyshev wavelet transform (CWT) is defined by

$$\left(\Im_{\psi} f\right)(b, a) \; = \; < f(t), \; \psi_{b,a}(t) >_w \tag{2.4}$$

$$. = \frac{1}{\pi}\int_{-1}^{1} f(t)\overline{\psi_{b,a}(t)} \; w(t)dt \tag{2.5}$$

$$= \frac{1}{\pi}\int_{-1}^{1} f(t)\overline{\tau_b \psi(at)} w(t)dt \tag{2.6}$$

provided the integral is convergent.
Since by Lemma 1.2 (i) and (2.2) $\psi_{b,a} \in X$ whenever $\psi \in X,$ by Theorem 1.3 the integral (2.6) is convergent for $f \in L^1_w(-1,1).$

Corresponding to the usual admissibility condition for the classical wavelet, we define the following admissibility condition for the Chebyshev wavelet,

$$A_{\psi} = \sum_{k=0}^{\infty}\frac{|\hat{\psi}(k)|^2}{k} < \infty. \tag{2.7}$$

From (2.7), we must have $\hat{\psi}(0) = 0.$ Note that

$$\hat{\psi}(k) = \frac{1}{\pi}\int_{-1}^{1}\psi(t)\ T_k(t)w(t)dt.$$

This and (2.7) together yield

$$\pi\hat{\psi}(0) = \int_{-1}^{1}\psi(t)\ T_0(t)w(t)dt = \int_{-1}^{1}\psi(t)w(t)dt = 0.$$

Hence, $\psi(t)$ changes sign in (-1,1). Therefore it represents a wavelet.

Theorem 2.1. If $\psi \in X$ defines a Chebyshev wavelet and $\phi \in L^1_w(-1,1),$ then the convolution $(\psi * \phi)$ defines a Chebyshev wavelet.

Proof. Let $\psi \in X$ and $\phi \in L^1_w(-1,1).$ Th en $\hat{\phi}$ is a bounded function on (-1,1). By Theorem 1.3, $\psi * \phi \in X.$ We have

$$A_{\psi*\phi} = \sum_k \frac{|(\psi*\phi)^\wedge(k)|^2}{k}$$

$$= \sum_k \frac{|\hat{\psi}(k)|^2|\hat{\phi}(k)|^2}{k}$$

$$\leq \|\phi\|_1^2\ \sum_k \frac{|\hat{\psi}(k)|^2}{k} < \infty.$$

Therefore, $(\psi * \phi)$ represents a Chebyshev wavelet.

Theorem 2.2 Let $f \in L^1_w(-1,1)$ and $\psi \in X$ and $(\Im_\psi f)(b,a)$ be the continuous Chebyshev wavelet transform. Then we have the following

$$\left\|(\Im_\psi f)(b,a)\right\|_X \leq \|f\|_1\ \ \|\psi\|_X .$$

Proof. The above inequality follows from (1.11).

3. A General Reconstruction Formula

In order to derive a general reconstruction formula, we need the following Lemma.

Lemma 3.1. Let ψ be a basic wavelet which defines continuous Chebyshev wavelet transform (2.6). Then

$$(\Im_\psi f)^\wedge(k,a) = \hat{f}(k)\ \overline{\hat{\psi}(a,k)}, \tag{3.1}$$

where

$$\hat{\psi}(a,k) = \frac{1}{\pi}\int_{-1}^{1}\overline{\psi(az)}\; T_k(z)w(z)dz. \tag{3.2}$$

Proof. From (2.6), we have

$$(\Im_\psi f)(b,a) = \frac{1}{\pi}\int_{-1}^{1} f(t)\overline{\tau_b\psi(at)}w(t)dt \tag{3.3}$$

$$= (f(t) * \overline{\psi(at)})\;(b). \tag{3.4}$$

From Theorem 1.3, it follows that

$$(\Im_\psi f)^\wedge(k,a) = \hat{f}(k)\overline{\hat{\psi}(a,k)},$$

which completes the proof of Lemma 3.1.

Theorem 3.2. Let ψ be a basic wavelet which defines Chebyshev wavelet transform (2.6) and $q(a) > 0$ be a weight function .
Assume that

$$Q(k) = \int_{0}^{1} q(a)\,|\,\hat{\psi}(a,k)\,|^2\; w(a)da > 0, \tag{3.5}$$

where $\hat{\psi}(a,k)$ is given by (3.2). Then

$$f(t) = \hat{f}(0) + \int_{0}^{1}\int_{-1}^{1} q(a)(\Im_\psi f)(b,a)\psi^{b,a}(t)w(a)w(b)\,da\,db, \tag{3.6}$$

where $\psi^{b,a}(t)$ is such that

$$\hat{\psi}^{b,a}(k) = \hat{\psi}_{b,a}(k)/Q(k). \tag{3.7}$$

Proof. From (3.1), we have

$$(\Im_\psi f)^\wedge(k,a) = \hat{f}(k)\overline{\hat{\psi}(a,k)};$$

so that

$$\frac{1}{\pi}\int_{-1}^{1}(\Im_\psi t)\;(b,a)\;T_k(b)w(b)db = \hat{f}(k)\overline{\hat{\psi}(a,k).}$$

Multiplying both sides by $\hat{\psi}(a,k)\,w(a)$ and the weight function $q(a) > 0$ and integrating both sides with respect to a from 0 to 1, we have

$$\frac{1}{\pi}\int_{0}^{1} q(a)\hat{\psi}(a,k)\left(\int_{-1}^{1}(\Im_\psi f)(b,a)T_k(b)w(b)db\right)w(a)da = \left[\int_{0}^{1} q(a)\,|\,\hat{\psi}(a,k)\,|^2\; w(a)da\right]\hat{f}(k) \tag{3.8}$$

Using (3.5), (3.8) gives

$$\hat{f}(k) = \frac{1}{\pi Q(k)}\int_{0}^{1} q(a)\;\hat{\psi}(a,k)\left(\int_{-1}^{1}(\Im_\psi f)(b,a)T_k(b)w(b)db\right)w(a)da$$

$$= \frac{1}{\pi Q(k)} \int_{0}^{1} q(a)\left(\int_{-1}^{1} (\Im_{\psi} f)(b,a)\,\hat{\psi}(a,k) T_k(b) w(b)\,db \right) w(a)\,da \tag{3.9}$$

We have from (2.2),

$$\psi_{b,a}(t) = \tau_b \psi(at).$$

From Lemma 1.2 (iv) it follows that

$$\hat{\psi}_{b,a}(k) = \hat{\psi}(a,k) T_k(b). \tag{3.10}$$

Using (3.10) in (3.9) we get

$$\hat{f}(k) = \frac{1}{\pi Q(k)} \int_{0}^{1} q(a) \int_{-1}^{1} (\Im_{\psi} f)(b,a)\,\hat{\psi}_{b,a}(k) w(a) w(b)\,da\,db$$

From (3.7) it follows that

$$\hat{f}(k) = \frac{1}{\pi} \int_{0}^{1} q(a) \int_{-1}^{1} (\Im_{\psi} f)(b,a)\,\hat{\psi}^{b,a}(k) w(a) w(b)\,da\,db. \tag{3.11}$$

Putting (3.11) in (1.8), we have

$$f(t) = \hat{f}(0) + \frac{2}{\pi} \sum_{k} T_k(t) \int_{0}^{1}\int_{-1}^{1} q(a) (\Im_{\psi} f)(b,a)\,\hat{\psi}^{b,a}(k) w(a) w(b)\,da\,db$$

$$= \hat{f}(0) + \frac{1}{\pi} \int_{0}^{1}\int_{-1}^{1} q(a) (\Im_{\psi} f)(b,a) \sum_{k} 2\,\hat{\psi}^{b,a}(k) T_k(t) w(a) w(b)\,da\,db$$

$$= \hat{f}(0) + \frac{1}{\pi} \int_{0}^{1}\int_{-1}^{1} q(a) (\Im_{\psi} f)(b,a) \psi^{b,a}(t) w(a) w(b)\,da\,db, \tag{3.12}$$

since $\hat{\psi}(0) = 0$. This completes the proof of Theorem 3.2.

Theorem 3.3. Assume that there exist positive constants A and B such that

$$0 < A \le Q(k) \le B < \infty, \quad \forall k \in \mathbf{N}_0. \tag{3.13}$$

Let $\psi^{a}(t)$ be a function defined by means of its Chebyshev transform.

$$\hat{\psi}^{a}(k) = \frac{\hat{\psi}(a,k)}{Q(k)}. \tag{3.14}$$

Then

(i) $\psi^{b,a}(t) = \tau_b \psi^{a}(t)$; (3.15)

(ii) $\| \psi^{b,a} \|_2 \le A^{-1} \| \psi_{b,a} \|_2$. (3.16)

Proof (i). In view of Lemma 1.2(iv)

$$\left(\tau_b\psi^a\right)^\wedge(k) = T_k(b)\overset{\wedge}{\psi}{}^a(k)$$

$$= \frac{T_k(b)\ \overset{\wedge}{\psi}(a,k)}{Q(k)}.$$

From (3.10) and (3.7), it follows that

$$\left(\tau_b\psi^a\right)^\wedge(k) = \overset{\wedge}{\psi}{}^{b,a}(k).$$

Taking inverse Chebyshev transform of both sides, we get

$$\tau_b\psi^a(t) = \psi^{b,a}(t).$$

(ii) From (3.7), we have

$$\left|\overset{\wedge}{\psi}{}^{b,a}(k)\right| \ \le\ A^{-1}\left|\overset{\wedge}{\psi}_{b,a}(k)\right|. \tag{3.17}$$

From (3.17) and (1.13), it follows that

$$\|\,\psi^{b,a}\,\|_2 \ \le\ A^{-1}\,\|\,\psi_{b,a}\,\|_2\,.$$

4. The Discrete Transform

The continuous Chebyshev wavelet transform of the function f in terms of two continuous parameters a and b can be converted into a semi-discrete Chebyshev wavelet transform by assuming that $a = 2^{-m}$; $m \in \mathbf{Z}$ and $-1 \le b \le 1$.

Now, we assume that $\psi \in L^2{}_w(-1,1)$ satisfies the so called “stability condition”

$$A \le \sum_{m=-\infty}^{\infty} |\,\overset{\wedge}{\psi}(2^{-m}k)\,|^2 \le B \tag{4.1}$$

for certain positive constants A and B, $0 < A \le B < \infty$. The function $\psi \in L^2{}_w(-1,1)$ satisfying (4.1) is called dyadic wavelet.

Using the definition (2.4), we define the semi-discrete Chebyshev wavelet transform of any $f \in L^2{}_w(-1,1)$ by

$$(\Im^{\psi}_{m} f)(b) = (\Im_\psi f)(b, \frac{1}{2^m}) = \ < f(t), \psi_{b,2^{-m}}(t) >_w \tag{4.2}$$

$$= \frac{1}{2}\int_{-1}^{1} f(t)\tau_b\overline{\psi(2^{-m}t)}w(t)dt \tag{4.3}$$

$$= (f * \overline{\psi}_m), \tag{4.4}$$

where

$$\psi_m(z) = \psi(2^{-m}z), \qquad m \in \mathbf{Z}.$$

Theorem 4.1. Assume that the semi-discrete CWT of any $f \in L^2{}_w(-1,1)$ is defined by (4.2).

Let us consider another wavelet ψ^* defined by means of its Chebyshev transform.

$$\hat{\psi}^*(k) = \frac{\hat{\psi}(k)}{\pi\sum_{j=-\infty}^{\infty}|\hat{\psi}(2^{-j}k)|^2}. \tag{4.5}$$

Then

$$f(t) = \hat{f}(0) + \sum_{m=-\infty}^{\infty}\int_{-1}^{1}\left(\Im_m^{\psi}f\right)(b)\left(\hat{\psi}^*(2^{-m}k)T_k(t)\right)^{\vee} w(b)db. \tag{4.6}$$

Proof. In view of (4.4), for any $f \in L^2{}_w(-1,1)$, we have

$$\begin{aligned}
&\hat{f}(0) + \sum_m\int_{-1}^{1}\left(\Im_m^{\psi}f\right)(b)\left(\hat{\psi}^*(2^{-m}k)T_k(t)\right)^{\vee} w(b)db \\
&= \hat{f}(0) + \sum_m\int_{-1}^{1}\left(\Im_m^{\psi}f\right)(b)\sum_k 2\,\hat{\psi}^*(2^{-m}k)T_k(t)T_k(b)w(b)db \\
&= \hat{f}(0) + \sum_m\sum_k 2\,\hat{\psi}^*(2^{-m}k)\,T_k(t)\int_{-1}^{1}\left(\Im_m^{\psi}f\right)(b)T_k(b)w(b)db \\
&= \hat{f}(0) + \pi\sum_m\sum_k 2\,\hat{\psi}^*(2^{-m}k)T_k(t)\left(\Im_m^{\psi}f\right)^{\wedge}(k) \\
&= \hat{f}(0) + \pi\sum_m\sum_k 2T_k(t)\,\hat{f}(k)\,\overline{\hat{\psi}(2^{-m}k)}\;\hat{\psi}^*(2^{-m}k) \\
&= \hat{f}(0) + \pi\sum_m\sum_k 2T_k(t)\,\hat{f}(k)\,\frac{\overline{\hat{\psi}(2^{-m}k)}\,\hat{\psi}(2^{-m}k)}{\pi\sum_j\left|\hat{\psi}(2^{-m}2^{-j}k\right|^2} \\
&= \hat{f}(0) + \sum_k 2\hat{f}(k)T_k(t) \\
&= f(t).
\end{aligned}$$

The above theorem leads to the following definition of dyadic dual.

Definition 4.2. A function $\tilde{\psi} \in L^2{}_w(-1,1)$ is called a dyadic dual of a dyadic wavelet ψ, if every $f \in L^2{}_w(-1,1)$ can be expressed as

$$f(t) = \hat{f}(0) + \sum_{m} \int_{-1}^{1} \left(\Im_m^{\psi} f\right)(b) \left(\hat{\hat{\psi}}(2^{-m}k) T_k(t) \right)^{\vee} (b) w(b) db. \tag{4.7}$$

So far we have considered semi-discrete Chebyshev wavelet transform of any $f \in L^2{}_w(-1,1)$ discretizing only variable a. Now, we discretize the translation parameter b also by restricting it to the discrete set of points

$$b_{m,n} = \frac{n}{2^m} b_0, \; m \in \mathbf{Z}, n \in \mathbf{N}_0, \tag{4.8}$$

where $b_0 \in [-1,1]$ is a fixed constant. We write

$$\psi_{b_0;m,n}(t) = \psi_{b_{m,n};a_m}(t) = \psi(2^{-m}t, 2^{-m}n \; b_0) \tag{4.9}$$

Then the discrete Chebyshev wavelet transform of any $f \in L^2{}_w(-1,1)$ can be expressed as

$$(\Im_{\psi} f) \; (b_{m,n;} a_m) = < f, \psi_{b_0;m,n} >_w \qquad m \in \mathbf{Z}, n \in \mathbf{N_0}. \tag{4.10}$$

The "stability" condition for this reconstruction takes the form

$$A \| f \|_2^2 \le \sum_{\substack{m \in Z \\ n \in N_0}} \left| < f, \psi_{b_0;m,n} >_w \right|^2 \le B \| f \|_2^2, \; f \in L^2{}_w(-1,1), \tag{4.11}$$

where A and B are positive constants such that $0 < A \le B < \infty$.

Theorem 4.3. Assume that the discrete CWT of any $f \in L^2{}_w(-1,1)$ is defined by (4.10) and stability condition (4.11) holds. Let T be a linear operator on $L^2{}_w(-1,1)$ defined by

$$Tf = \sum_{\substack{m \in Z \\ n \in N_0}} < f, \psi_{b_0;m,n} >_w \psi_{b_0;m,n}. \tag{4.12}$$

Then

$$f = \sum < f, \psi_{b_0;m,n} >_w \; \psi_{b_0}^{m,n}, \tag{4.13}$$

where

$$\psi_{b_0}^{m,n} = T^{-1} \psi_{b_0;m,n}; \quad m \in \mathbf{Z}, n \in \mathbf{N}_0.$$

Proof. From the stability condition (4.11), it follows that the operator defined by (4.12) is a one-one bounded linear operator.

Set

$$g = Tf, \quad f \in L^2{}_w(-1,1).$$

Then, we have

$$< Tf, f > = \sum_{\substack{m \in Z \\ n \in N_0}} \left| < f, \psi_{b_0;m,n} >_w \right|^2 .$$

Therefore

$$A \| T^{-1} g \|_2^2 = A \| f \|_2^2 \le < Tf, f >_w = < g, T^{-1} g >_w \le \| g \|_2 \| T^{-1} g \|_2 ;$$

so that

$$\| T^{-1} g \|_2 \le \frac{1}{A} \| g \|_2 .$$

Hence, every $f \in L^2{}_w(-1,1)$ can be reconstructed from its discrete CWT given by (4.10). Thus

$$f = T^{-1} T f = \sum_{\substack{m \in Z \\ n \in N_0}} < f, \psi_{b_0;m,n} >_w T^{-1} \psi_{b_0;m,n} . \tag{4.14}$$

Finally, set

$$\psi_{b_0}^{m,n} = T^{-1} \psi_{b_0;m,n} ; \quad m \in \mathbf{Z}, \quad n \in \mathbf{N}_0.$$

Then, the reconstruction (4.14) can be expressed as

$$f = \sum_{\substack{m \in Z \\ n \in N_0}} < f, \psi_{b_0;m,n} >_w \psi_{b_0}^{m,n} ,$$

which completes the proof of Theorem 4.3.

5. Frames and Riesz basis in $L^2{}_w(-1,1)$.

In this section, using $\psi_{b_0;m,n}$, a frame is defined and Riesz basis of $L^2{}_w(-1,1)$ is studied.

Definition 5.1. A function $\psi \in L^2{}_w(-1,1)$ is said to generate a frame $\{\psi_{b_0;m,n}\}$ of $L^2{}_w(-1,1)$ with sampling rate b_0 if (4.11) holds for some positive constants A and B. If A = B, then the frame is called a tight frame.

Definition 5.2. A function $\psi \in L^2{}_w(-1,1)$ is said to generate a Riesz basis of $\{\psi_{b_0;m,n}\}$ with sampling rate b_0 if the following two properties are satisfied.

(i) The linear span $< \psi_{b_0;m,n} : m \in \mathbf{Z} >$ is dense in $L^2{}_w(-1,1)$ (5.1)

(ii) There exist positive constants A and B with $0 < A \le B < \infty$ such that

$$A \| \{c_{m,n}\} \|_2^2 \le \left\| \sum_{\substack{m \in Z \\ n \in N_0}} c_{m,n} \psi_{b_0;m,n} \right\|_2^2 \le B \left\| \{c_{m,n}\} \right\|_{\ell^2}^2 \tag{5.2}$$

for all $\{c_{m,n}\} \in \ell^2(\mathbf{N}_0^2)$. Here A and B are called the Riesz bounds of $\{\psi_{b_0;m,n}\}$.

Theorem 5.3. Let $\psi \in L^2{}_w(-1,1)$, then the following statements are equivalent.

(i) $\{\psi_{b_0;m,n}\}$ is a Riesz basis of $L^2{}_w(-1,1)$;

(ii) $\{\psi_{b_0;m,n}\}$ is a frame of $L^2{}_w(-1,1)$ and is also an ℓ^2 - linearly independent family in the sense that if $\sum \psi_{b_0;m,n} c_{m,n} = 0$ and $\{c_{m,n}\} \in \ell^2$, then $c_{m,n} = 0$. Furthermore, the Riesz bounds and frame bounds agree.

Proof. It follows from (5.2) that any Riesz basis is ℓ^2-linearly independent.
Let $\{\psi_{b_0;m,n}\}$ be a Riesz basis with Reisz bounds A and B, and consider the matrix operator

$$M = [\gamma_{r,s,m,n}]_{(r,s),(m,n)\in N_0 \times N_0}$$

where the entries are defined by

$$\gamma_{r,s,m,n} = \langle \psi_{b_0;r,s}, \psi_{b_0;m,n} \rangle . \tag{5.3}$$

Then from (5.2), we have

$$A \| \{c_{m,n}\} \|^2_{\ell^2} \le \sum_{r,s,m,n} c_{r,s} \gamma_{r,s,m,n} c_{m,n} \le B \| \{c_{m,n}\} \|^2_2 ;$$

so that M is positive definite. We denote the inverse of M by

$$M^{-1} = [\mu_{r,s,m,n}]_{(r,s),(m,n)\in N_0^2} , \tag{5.4}$$

which means that both

$$\sum_{t,u} \mu_{r,s;t,u} \gamma_{t,u;m,n} = \delta_{r,m} \delta_{s,m;} \ r, s, m, n \in \mathbf{N}_0 \tag{5.5}$$

and

$$B^{-1} \left\| \{c_{m,n}\} \right\|^2_{\ell^2} \le \sum_{r,s,m,n} c_{r,s} \ \mu_{r,s,m,n} \ c_{m,n} \le A^{-1} \left\| \{c_{m,n}\} \right\|^2_{\ell^2} \tag{5.6}$$

are satisfied. This allows us to introduce

$$\psi^{r,s}(x) = \sum_{m,n} \mu_{r,s;m,n} \psi_{b_0;m,n}(x) . \tag{5.7}$$

Clearly, $\psi^{r,s} \in L^2_w(-1,1)$ and it follows from (5.3) and (5.5) that

$$\left\langle \psi^{r,s} ; \psi_{b_0;m,n} \right\rangle_w = \delta_{r,m} \delta_{s,n} ; \ r, s, m, n \in \mathbf{N}_0,$$

which means that $\{\psi^{r,s}\}$ is the basis of $L^2{}_w(-1,1)$, which is dual to $\{\psi_{b_0;m,n}\}$.

Furthermore, from (5.5) and (5.6), we conclude that

$$\left\langle \psi^{r,s}, \psi^{m,n} \right\rangle_w = \mu_{r,s,m,n}$$

and the Riesz bounds of $\{\psi^{r,s}\}$ are B^{-1} and A^{-1}.
In particular, for any $f \in L^2{}_w(-1,1)$ we may write

$$f(x) = \sum_{m,n} < f, \psi_{b_0;m,n} >_w \ \psi^{m,n}(x)$$

$$\text{and} \quad B^{-1} \sum_{m,n} \left| < f, \psi_{b_0;m,n} >_w \right|^2 \le \| f \|_2^2 \le A^{-1} \sum_{m,n} \left| < f, \psi_{b_0;m,n} >_w \right|^2 . \qquad (5.8)$$

Since, (5.8) is equivalent to (4.11), therefore, statement
(i) implies statement (ii). To prove the converse part, we recall Theorem 4.3 and we have for any $g \in L^2{}_w(-1,1)$ and f = T^{-1}g,

$$g(x) = \sum_{\substack{m \in Z \\ n \in N_0}} < f, \psi_{b_0;m,n} >_w \ \psi_{b_0;m,n} .$$

Also, by the ℓ^2-linear independence of $\{\psi_{b_0;m,n}\}$, this representation is unique. From the Banach-Steinhaus and open mapping theorem it follows that $\{\psi_{b_0;m,n}\}$ is a Riesz basis of $L^2_w(-1,1)$.

6. Example of Chebyshev Wavelet

In this section the Chebyshev wavelet corresponding to the function $\psi(t) = t^3 \ ; t \in [-1,1]$ is given.
Using representation (2.3), we have

$$\psi_{b,a}(t) = \frac{1}{2}\left\{\psi(abt + \left[(1-a^2t^2)(1-b^2)\right]^{\frac{1}{2}}) + \psi(abt - [(1-a^2t^2)(1-b^2)]^{1/2})\right\}$$

$$= \frac{1}{2}\left\{(abt + [(1-a^2t^2)(1-b^2)]^{1/2})^3 + (abt - [(1-a^2t^2)(1-b^2)]^{1/2})^3\right\}$$

$$= a^3b^3t^3 + 3ab(1-b^2)(t - a^2t^3). \qquad (6.1)$$

7. Example of Chebyshev wavelet transform

In this section an example of Chebyshev wavelet transform is given.
Let $f(t) = t; \ \psi(t) = t^3; t \in [-1,1]$.
Now, using representation (2.5) the Chebyshev wavelet transform of the function f(t) with respect to the wavelet $\psi_{b,a}(t)$ given by (6.1) obtained as follows:

$$(\Im_\psi f)(b,a) = \frac{1}{\pi}\int_{-1}^{1} f(t)\psi_{b,a}(t)w(t)dt$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{\left[a^3b^3t^4 + 3ab(1-b^2)(t^2 - a^2t^4)\right]}{\sqrt{1-t^2}} dt$$

$$= \frac{1}{\pi}\left[a^3b^3 \int_{-1}^{1} \frac{t^4}{\sqrt{1-t^2}} dt + 3ab(1-b^2) \int_{-1}^{1} \frac{t^2}{\sqrt{1-t^2}} dt - 3a^3b(1-b^2) \frac{t^4}{\sqrt{1-t^2}} dt \right]$$

$$= \frac{2}{\pi}\left[a^3b(4b^2-3) \int_{0}^{1} \frac{t^4}{\sqrt{1-t^2}} dt + 3ab(1-b^2) \int_{0}^{1} \frac{t^2}{\sqrt{1-t^2}} dt \right].$$

Making the substitution t = sin θ and evaluating the above integral, we have

$$(\Im_\psi f)(b,a) = \frac{3}{8} ba^3(4b^2-3) + \frac{3}{2} ab(1-b^2).$$

8. Order of Approximation

In this section following the technique of Depczynski [5] a discussion on the expansion of $f \in L^2_w(-1,1)$ in wavelet series is given and order of wavelet coefficient is obtained.

Let N_m, $m \in \mathbf{N}$, be a strictly increasing sequence of natural numbers and denote by V_m the space

$$V_m = \text{span } \{f_i \ : \ i \in \Delta\}, \tag{8.1}$$

where $\Delta = \{ 1, ---, N_m \}$ is any index set and $\{f_i\}$ is the basis of Hilbert space $L^2_w(-1,1)$.

The spaces V_m are linear and closed subspaces of $L^2_w(-1,1)$ and $V_m \subset V_{m+1}$. Moreover,

$$\overline{\bigcup_{m=1}^{\infty} V_m} = L^2_w(-1,1). \tag{8.2}$$

The orthogonal complement of V_m in V_{m+1} is denoted by W_m, i.e. we have

$$V_{m+1} = V_m \oplus W_m \tag{8.3}$$

From (8.2) and (8.3), we have

$$V_1 \oplus W_1 \oplus W_2 \oplus --- = L^2_w(-1,1). \tag{8.4}$$

Now, we study the approximation properties of the spaces V_m.

Let us consider the Chebyshev differential equation

$$\left(\sqrt{1-x^2} f'\right)' = \frac{-n^2 f}{\sqrt{1-x^2}} \tag{8.5}$$

together with the following homogeneous boundary conditions

$$U_{-1}(f) = a_1 f(-1) + a_2 f(1) = 0,$$

$$U_1(f) = b_1 f(-1) + b_2 f(1) = 0.$$

The eigenvalues of the above boundary value problem are given by $\lambda_n = n^2$, $n \in \mathbf{N}$.

Let T_0, T_1, T_2, --- be the corresponding eigenfunctions.

Now, we introduce the space

$$D = \left\{ f \in W_2^2[-1,1]; U_{-1}(f) = 0 \quad \text{and} \quad U_1(f) = 0 \right\}$$

where $W_2^2[-1,1] = \left\{ f \in C'[-1,1]; f'' \in L_w^2(-1,1) \right\}$.

We will always assume that zero is no eigenvalue of (8.5). From [5.p 236] we know that the Green's function g(x,.y) of (8.5) is given by

$$g(x,y) = \sum_{n=0}^{\infty} \frac{T_n(x)\overline{T_n(y)}}{\lambda^n}, \quad (x,y) \in [-1,1]\times[-1,1]$$

and $G : L_w^2(-1,1) \longrightarrow L_w^2(-1,1)$ defined by

$$G(f) = \int_{-1}^{1} g(x,y)\, f(y)\, w(y)\, dy$$

is a compact operator with range D.

For $r \in \mathbf{N}$, set

$$G^r = G \circ --- \circ G (r \text{ times})$$

The iterated spaces D^r, $r \in \mathbf{N}_0$, are then defined by

$$D^r = \left\{ G^r(f) : f \in L_w^2(-1,1) \right\}.$$

Note that $D^0 = L_w^2(-1,1)$ and $D^1 = D$ and $D^r \supset D^{r+1}$.

From [5, p. 237], we also note that

$$f \in D^r \Leftrightarrow \sum_{n=0}^{\infty} \lambda_n^{2\lambda} |< f, T_n >_w|^2 < \infty \tag{8.6}$$

We recall the following result from [5, p.237] about approximation with the spaces V_m = span $\{T_1, T_2 ----T_N\}$, $N_m < N_{m+1}$.

Theorem 8.1. Let $P_m f = \sum_{n=0}^{N_m} < f, T_n >_w T_n$ denote the orthogonal projection of f onto V_m. Then for $r \in \mathbf{N}$ and $f \in D^r$,

$$\| f - P_m f \|_2 = O\left(N_m^{-2r+\frac{1}{2}} \right), m \longrightarrow \infty. \tag{8.7}$$

Let $\psi_{b_0;m,n}$ be the discrete Chebyshev wavelets defined by (4.9). Then for any $f \in L_w^2(-1,1)$, there exists a sequence $\{c_{m,n}\}$ such that

$$Q_m f = \sum_{n=1}^{M_m} c_{m,n} \;\; \psi_{b_0;m,n}\,, \tag{8.8}$$

where $M_m = N_{m+1} - N_m$.

Suppose $Q_m : L_w^2(-1,1) \longrightarrow L_w^2(-1,1)$ be the operator such that $Q_m = P_{m+1} - P_m$, where P_m is the projection operator defined in Theorem 8.1. The range $Q_m = W_m$, where $V_{m+1} = V_m \oplus W_m$.

Following Chui [4], we have the following definition for the Riesz basis of the spaces W_m which will be used in sequel.
The discrete Chebyshev wavelet $\psi_{b_0;m,n}$ forms a Riesz basis of the spaces W_m, if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\sum_n |c_{m,n}|^2 \le \left\| \sum_n c_{m,n}\psi_{b_0;m,n} \right\|_2^2 \le B\sum_n |c_{m,n}|^2 \tag{8.9}$$

for all sequences $\{c_{m,n}\}$ such that $\sum_n |c_{m,n}|^2 < \infty$.

Theorem 8.2. Let $\{\psi_{b0;m,n}\}$ be a Riesz basis of the spaces W_m with Riesz bound $A > 0$. Then for every $r \in \mathbf{N}$ and $f \in D^r$,

$$\left(\sum_n |c_{m,n}|^2\right)^{1/2} = O\left(N_m^{-2r+\frac{1}{2}}\right),\ m \longrightarrow \infty. \tag{8.10}$$

Proof. $\|Q_m f\|_2 = \| P_{m+1}f - P_m f \|_2 \le \| f - P_m f \|_2 + \| f - P_{m+1} f \|_2$
From (8.7), it follows that

$$\| Q_m f \|_2 = O\left(N_m^{-2r+\frac{1}{2}}\right). \tag{8.11}$$

Using the Riesz stability condition (8.9) and relation (8.8), we have

$$\| Q_m f \|_2 = \left\| \sum_n c_{m,n}\psi_{b_0;m,n} \right\|_2^2 \ge A\sum_n |c_{m,n}|^2;$$

so that

$$\sum_n |c_{m,n}|^2 \le \frac{1}{A} \| Q_m f \|_2^2 .$$

From (8.11), it follows that

$$\left(\sum_n |c_{m,n}|^2\right)^{1/2} = O\left(N_m^{-2r+\frac{1}{2}}\right), \tag{8.12}$$

which completes the proof of Theorem 8.2.

Theorem 8.3. Let $\psi_{b_0;m,n}$ be a Riesz basis of the spaces W_m with upper Riesz bound $B<\infty$. Assume that $\sum_{m=1}^{\infty} N_m^{-2\epsilon} < \infty$ for some $\epsilon > 0$. If $r \in \mathbf{N}$ and $f \in L_w^2(-1,1)$ with $\left(\sum_n |c_{m,n}|^2\right)^{1/2} = O\left(N_m^{-2r-\epsilon}\right), n \longrightarrow \infty,$ then $f \in D^r$.

Proof. Let us consider the partial sum $\sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |< f, T_n >_w|^2$ of series $\sum_{n=0}^{\infty} \lambda_n{}^{2r} |< f, T_n >_w|^2$.

For $\lambda_n = n^2$, we obtain

$$\sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |< f, T_n >_w|^2 \leq N_{m+1}^{4r} \sum_{n=N_m+1}^{N_{m+1}} |< f, T_n >_w|^2 \, . \tag{8.13}$$

Now, $Q_m f = P_{m+1} - P_m f$.

Therefore

$$\| Q_m f \|_2^2 = \| P_{m+1} f - P_m f \|_2^2$$

$$. = \sum_{n=N_m+1}^{N_{m+1}} |< f, T_n >_w|^2 \qquad . \tag{8.14}$$

Using the Riesz stability condition (8.9), we have

$$\sum_{n=N_m+1}^{N_{m+1}} |< f, T_n >_w|^2 = \| Q_m f \|_2^2 \leq B \sum_n | c_{m,n} |^2 \tag{8.15}$$

Next using the assumption $\left(\sum_n | c_{m,n} |^2 \right)^{1/2} \leq C_2 N_m^{-2r-\in}$, from (8.13), (8.14) and (8.15), it follows that

$$\sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |< f, T_n >_w|^2 \leq N_{m+1}^{4r} C_2^2 \; N_m^{-4r-2\in} = C_2^2 B \left(\frac{N_{m+1}}{N_m} \right)^{4r} N_m^{-2\in} \, .$$

Let C>0 with $C_2^2 \; B \left(\frac{N_{m+1}}{N_m} \right)^{4r} < C$ for all $m \in \mathbf{N}$.

Then from the covergence of $\sum_{m=1}^{\infty} N_m^{-2\in} < \infty$, it follows that

$$\sum_{m=1}^{\infty} \sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |< f, T_n >_w|^2 \leq C \sum_{m=1}^{\infty} N_m^{-2\in} < \infty \, .$$

From (8.6) it easily follows that $f \in D^r$.
This complete the proof of Theorem 8.3

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