

Unchanging Independent Domination in Graphs

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Abstract

A dominating set $D \subseteq V(G)$ is said to be an independent dominating set if $\langle D \rangle$ has no edges. The minimum cardinality of an independent dominating set is denoted by $i(G)$. In this paper, we examine the effects on $i(G)$ when G is modified by deleting a vertex or an edge or adding an edge.

Key words: Domination, Domination number, independent domination, independent domination number.

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1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretical terms we refer to Harary [2] and for terms related to domination we refer to Haynes et al.[4]. A subset D of $V(G)$ is said to be a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D . A dominating set $D \subseteq V(G)$ is said to be an independent dominating set if $\langle D \rangle$ has no edges. The minimum cardinality of an independent dominating set is denoted by $i(G)$. For any graph theoretic parameter, the study of determining the effect of removal of an edge or a vertex from the graph has several important applications such as fault tolerance in networks. The behaviour of a network in the presence of a fault can be analyzed by determining the effect that removing an edge (link failure) or a vertex (processor failure) from its underlying graph G has on the fault-tolerance criterion. For example, an $i(G)$ -set in G represents a minimum set of processors that can communicate directly with all other processors in the system and which have no linkage among themselves. It is important that $i(G)$ does

not increase when G is modified by removing a vertex or an edge. Carrington et al.[1] surveyed the problems involved in changing the domination number of a graph. A detailed study is given in Chapter 5 of Haynes et al.[4]. Here we examine the effects on $i(G)$ when G is modified by deleting a vertex or an edge or adding an edge.

2. TERMINOLOGY

Let $G - v$ (respectively $G - e$, $G + e$) denote the graph formed by removing vertex v (respectively edge e , adding an edge e) from G . We use acronyms to denote the following classes of graphs (U represents unchanging ; V : vertex; E : edge; R : removal; A : addition)

$$(UVR) \ i(G - v) = i(G) \text{ for all } v \in V$$

$$(UER) \ i(G - e) = i(G) \text{ for all } e \in E$$

$$(UEA) \ i(G + e) = i(G) \text{ for all } e \in E(\bar{G})$$

The vertex set of G is partitioned into three sets according to how the removal of vertices affect $i(G)$.

Let $V = V^0 \cup V^+ \cup V^-$ where

$$V^0 = \{v \in V : i(G - v) = i(G)\}$$

$$V^+ = \{v \in V : i(G - v) > i(G)\} \text{ and}$$

$$V^- = \{v \in V : i(G - v) < i(G)\}.$$

Similarly, the edge set can be partitioned into

$$E^0 = \{e \in E : i(G - e) = i(G)\} \text{ and}$$

$$E^+ = \{e \in E : i(G - e) > i(G)\}.$$

For example , consider the graph given in the Figure 1.

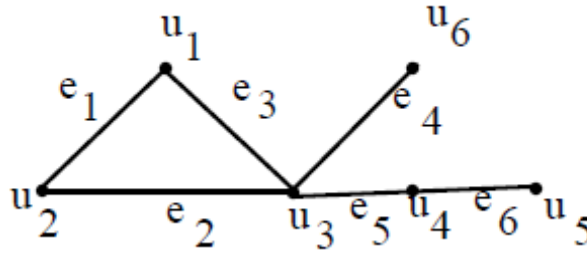


Figure 1

Here $V^0 = \{u_1, u_2, u_4, u_6\}$, $V^+ = \{u_3\}$,

$V^- = \{u_5\}$, $E_0 = \{e_1, e_5, e_6\}$ and

$E^+ = \{e_2, e_3, e_4\}$.

3. UNCHANGING VERTEX REMOVAL (U V R)

Example 3.1

1. If $G \cong K_p$ then $V^0 = V$ and so $G \in UV R$.
2. If G is a unicyclic graph obtained by drawing an edge between a vertex of a cycle C_{3k} and a pendent vertex of a path P_{3l} then $G \in UV R$.

Lemma 3.2 Let G be any graph with $\delta(G) = 1$. If $V^0 = V$, then every support is adjacent to exactly one pendent vertex.

Proof. Suppose u is a support which is adjacent to two or more pendent vertices. If there exists an $i(G)$ - set containing u , removal of u increases $i(G)$. If not, removal of any pendent vertex adjacent to u decreases $i(G)$. These contradictions prove that every support is adjacent to exactly one pendent vertex.

Remark 3.3 Converse of Lemma 3.2 is not true. If $G \cong P_4$ then $V^0(G) \neq V(G)$.

Definition 3.4 Two supports u and v are said to be consecutive if the unique $u - v$ path contains no other support.

Theorem 3.5 Let T be a tree. Then $V(T) = V^0$ if and only if every support is adjacent to exactly one pendent vertex and $d(u, v) \equiv 2 \pmod{3}$ for any two consecutive supports u and v .

Proof. Suppose that every support is adjacent to exactly one pendent vertex and $d(u, v) \equiv 2 \pmod{3}$ for any two consecutive supports u and v . Let u and v be two consecutive supports. Let $u = u_1, u_2, \dots, u_n = v$ be the $u - v$ path where $n \equiv 0 \pmod{3}$. Let S be any $i(T)$ - set. Without loss of generality we can assume that $\{u = u_1, u_4, u_7, \dots, v\} \subseteq S$. For $u_i \in S$ with $1 < i < n$, $S - \{u_i\}$ does not dominate u_{i-1} . For $u_i, i = 1$ to n , $S - \{u_i\}$ does not dominate v_i , the pendent adjacent to u_i . Removal of any other vertex in T does not change $i(T)$. Hence $V^0 = V$. Conversely, assume $V^0 = V$. By Lemma 3.2, every support is adjacent to exactly one pendent vertex. Suppose there exists two consecutive supports u and v such that $d(u, v) \not\equiv 2 \pmod{3}$.

Case(i). $d(u, v) \equiv 0 \pmod{3}$

Let $u = u_1, u_2, \dots, u_n = v$ be the $u - v$ path where $n \equiv 1 \pmod{3}$. Without loss of generality we can assume that if S is any $i(T)$ - set then $\{u_1, u_4, \dots, u_n\} \subseteq S$. Now $S - \{u_4\}$ increases $i(G)$ since now u_3 is not dominated by this set. Hence $u_4 \notin V^0$ which is a contradiction.

Case(ii). $d(u, v) \equiv 1 \pmod{3}$

Let u_1, v_1 be the pendent vertices adjacent to u and v . Let S be any $i(T)$ - set. Since u, v are adjacent, without loss of generality, we can assume that $\{u_1, v_1\} \subseteq S$. Now $S - \{v_1\}$ decreases $i(T)$ and so $v_1 \notin V^0$ which is a contradiction. Thus $d(u, v) \equiv 2 \pmod{3}$ for every two consecutive supports u and v .

Corollary 3.6 If $G \cong P_p$ then $V^0 = V$ if and only if $p = 3k + 2 (k \geq 0)$.

Theorem 3.7 For any tree T with at least two vertices, $V^0 \neq \varphi$.

Proof. Let u be a pendent vertex and v be the support adjacent to u . Every $i(T)$ - set S of T contains either u or v . If $u \in S$, $v \in V^0$ and if $v \in S$, $u \in V^0$. Thus $V^0 \neq \varphi$.

4. UNCHANGING EDGE REMOVAL (UER)

Example 4.1

(i) If $G \cong C_p$ then $E^0 = E$ and so then $G \in UER$.

(ii) If $G \cong K_p$ then $G \in UER$.

(iii) If $G \cong K_{m,n} (m, n \geq 2)$ then $G \in UER$.

Theorem 4.2 If P_p is a path on p vertices where $p = 3k + 1 (k \geq 0)$ then $E^0 = E$.

Proof. Let $P_p = (1, 2, \dots, p)$ and let e_1, e_2, \dots, e_{3k} be the edges of P_p . Then $i(P_p) = k + 1$. Consider $P_p - e_j (1 \leq j \leq 3k)$. Let $P_p - e_j = P_1 \cup P_2$ where $P_1 = (1, 2, \dots, j)$ is a path on j vertices and $P_2 = (j + 1, j + 2, \dots, 3k + 1)$ is a path on $3k + 1 - j$ vertices.

Case(i). $j \equiv 0(mod 3)$

Now $3k + 1 - j \equiv 1(mod 3)$ and so

$$i(P_p - e_j) = i(P_1) + i(P_2) = \left\lfloor \frac{j}{3} \right\rfloor + \left\lfloor \frac{3k+1-j}{3} \right\rfloor = \frac{j}{3} + \frac{3k+1-j-1}{3} + 1 = \frac{3k+3}{3} = k + 1 = i(P_p).$$

Case(ii). $j \equiv 1(mod 3)$

Now $3k + 1 - j \equiv 0(mod 3)$ and so

$$i(P_p - e_j) = i(P_1) + i(P_2) = \left\lfloor \frac{j}{3} \right\rfloor + \left\lfloor \frac{3k+1-j}{3} \right\rfloor = \frac{j-1}{3} + 1 + \frac{3k+1-j}{3} = \frac{3k+3}{3} = k + 1 = i(P_p).$$

Case(iii). $j \equiv 2(mod 3)$

Now $3k + 1 - j \equiv 2(mod 3)$ and so

$$i(P_p - e_j) = i(P_1) + i(P_2) = \left\lfloor \frac{j-2}{3} \right\rfloor + 1 + \left\lfloor \frac{3k+1-j-2}{3} \right\rfloor + 1 = k + 1 = i(P_p) .$$

Hence $E^0 = E$.

Proposition 4.3 For any connected graph G , $G \circ K_1 \in UER$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_p\}$ and $E(G) = \{e_1, e_2, \dots, e_p\}$. If $S = \{u_i, 1 \leq i \leq p\}, \{d_i, 1 \leq i \leq p\}$ are the sets of pendent vertices and pendent edges of $G \circ K_1$ respectively, then S is a minimum independent dominating set of $G \circ K_1$

and so $i(G \circ K_1) = p$. Also $i(G \circ K_1 - e_i) = p$ for every $1 \leq i \leq q$. Consider $(G \circ K_1) - d_j$ where $d_j = v_j u_j$. Since G is connected, there exists a vertex $v_l \in V(G)$ such that v_l is adjacent to v_j . Now $S = \{u_l\} \cup \{v_l\}$ is an $i(G)$ -set of $G \circ K_1$ and so $i((G \circ K_1) - d_j) = p$ for all $1 \leq j \leq p$. Thus $G \circ K_1 \in UER$.

5. UNCHANGING EDGE ADDITION (UEA)

Example 5.1

(i) If $G \cong mK_2$ then $G \in (CER) \cap (UEA)$.

(ii) If $G \cong K_{1,p-1}$ then $G \in (CER) \cap (UEA)$.

(iii) If $G \cong K_p - e$ then $G \in UEA$.

Theorem 5.2 If P_p is a path on p vertices, then $P_p \in UEA$ if and only if $p = 3$ or $p = 3k + 2(k \geq 1)$.

Proof. Suppose $P_p \in UEA$. Let $p = 3k$. Let $P_p = (1, 2, \dots, 3k)$ and suppose that $k > 1$. Consider the edge $e = (2, 5)$. Then $i(P_p + e) = k + 1$ whereas $i(P_p) = k$ and so $P_p \notin UEA$ which is a contradiction. Hence $p \neq 3k (k > 2)$. Similarly if $p = 3k + 1$ with $e = (1, 3)$, $i(P_p + e) = k$ whereas $i(P_p) = k + 1$ and so $P_p \notin UEA$ which is a contradiction. Hence $n \neq 3k + 1$ and so either $p = 3$ or $p = 3k + 2$.

Conversely, assume $p = 3k + 2(k \geq 1)$. Then $i(P_p) = k + 1$. We prove that $P_p \in UEA$ by induction on k . When $k = 1, p = 5$ and one can verify that $i(P_5 + e) = i(P_5) = 2$ for all $e \in E(\bar{G})$. Assume that the result is true for k . Now, we prove the result for $p = 3(k + 1) + 2 = 3k + 5$. Let $P_{3k+5} = (1, 2, \dots, 3k + 5)$. By induction hypothesis, any edge joining two vertices among $\{1, 2, \dots, 3k + 2\}$ will not change $i(G)$. On adding $e = (3k + 3, 3k + 5)$, we observe that $i(G + e) = k + 2$. Also it is easy to verify that any edge joining a vertex of $\{1, 2, \dots, 3k + 2\}$ and a vertex of $\{3k + 3, 3k + 4, 3k + 5\}$ does not change $i(G)$. Hence $P_p \in UEA$.

Proposition 5.3 A cycle $C_p \in UEA$ if and only if $p \not\equiv 1 \pmod{3}$.

Proof. Suppose $C_p \in UEA$ and $p \equiv 1 \pmod{3}$. Let $p = 3k + 1$ and $C_p = (v_1, v_2, \dots, v_{3k+1})$. Then $i(C_p + e) = k$ where $e = (v_1, v_3)$ and $i(C_p) = k + 1$. This violation of the assumption helps to conclude that $p \not\equiv 1 \pmod{3}$. Conversely, if $p \equiv 0 \pmod{3}$ or $p \equiv 2 \pmod{3}$, it is easy to observe that $C_p \in UEA$.

Proposition 5.4 Let T be a caterpillar in which alternate vertices are supports and at most one support is adjacent to two or more pendent vertices. Then $T \in UEA$.

Proof. By choice of T , the set of all supports S is a minimum independent dominating set. Suppose an edge is drawn between two supports u and v where u is one which is adjacent to exactly one pendent vertex. Then $S - \{u\} \cup \{w\}$ is a minimum independent dominating set where w is the pendent vertex adjacent to u . Addition of any other edge leaves S unaffected and so $T \in UEA$.

Remark 5.5 If the caterpillar described in Proposition 5.4 has two or more supports which are adjacent to two or more pendent vertices, addition of an edge between these supports increases $i(G)$ and so $T \notin UEA$.

The hypothesis of Proposition 5.4 is not necessary. For example, $P_5 \in UEA$ but P_5 does not satisfy the hypothesis of Proposition 5.4.

Proposition 5.6 If V^- is empty then $G \in UEA$.

Proof. Suppose $G \in UEA$ and $\in V^-$. Then $i(G - v) < i(G)$ and let S be an $i(G)$ - set of $G - v$. Clearly $N(v) \cap S = \emptyset$. Now adding an edge $e = (v, u)$ for any $u \in S$ we have $i(G + e) < i(G)$ which is a contradiction. Hence V^- is empty.

Remark 5.7 Converse of Proposition 5.6 is not true.

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