On Einstein Nearly Kenmotsu Manifolds

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Abstract

The present paper deals with the study of Einstein nearly Kenmotsu manifold with projective curvature tensor $P$ and conharmonic curvature tensor $N$ satisfying $R(X,Y), P = 0$ and $R(X,Y), N = 0$ and have shown manifold satisfying these condition is locally isometric to hyperbolic space $H^n(-1)$.

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1. INTRODUCTION

Tanno, S. ([10]) Classified connected almost metric manifold whose automorphism group possess the maximum dimension. for such a manifold $M$, the sectional curvature of the plane section $\xi$ is constant say $c$. If $c>0$, $M$ is a homogeneous Sasakian manifold of constant sectional curvature. If $c =0$, $M$ is a product of line or a circle with a Kaehler manifold of constant holomorphic section curvature. If $c<0$, $M$ is wrapped product space $R \times f C^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifold satisfying some special condition and characterized the differential Geometric properties of the manifolds of class(3); the structure so obtained is now known as Kenmotsu structure.

An almost contact manifold $(M, \phi, \xi, \eta, g)$ is called nearly kenmotsu manifold by Shukla, A. ([9]) if the following relations holds:

(1.1) $\nabla_X \phi + (\nabla_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y$

Where $\nabla$ is Levi-Civita connection of $g$. Moreover, if $M$ satisfies
then it is called a Kenmotsu manifold([4]). It is easy to see that every Kenmotsu manifold is nearly Kenmotsu manifold but converse is not true. Many other author ([6, 7, 8]) have studied nearly Kenmotsu manifold briefly.

In this paper we have study Einstein nearly Kenmotsu manifold with projective curvature tensor P and conharmonic curvature tensor N and have proved four interesting theorems.

2. ON NEARLY KENMOTSU MANIFOLDS

An \( n \) dimensional differentiable manifold \( M \) is called an almost contact Riemannian manifold if, there is an almost contact structure \( (\varphi, \xi, \eta) \) consisting of a \((1, 1)\) tensor field \( \varphi \), a vector field \( \xi \) and 1-form \( \xi \) satisfying ([6, 7, 8])

\[
\begin{align*}
\varphi^2(X) &= -X + \eta(X)\xi \\
\eta(\xi) &= 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0
\end{align*}
\]

Let \( g \) be Riemannian metric with \((\varphi, \xi, \eta)\), that is,

\[
\begin{align*}
g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X) \eta(Y) \\
\text{Or equivalently} \quad g(X, \varphi Y) &= -g(\varphi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X)
\end{align*}
\]

For all vector fields \( X, Y, M \) on \( M \). If an almost contact metric manifold satisfies (1.1) then \( M \) is called nearly Kenmotsu manifold.

In every \( n \) –dimensional nearly Kenmotsu manifold \((M, \varphi, \xi, \eta, g)\) the following important identities holds. (for more detail see ([7, 8]))

\[
\begin{align*}
(\nabla_X \eta)Y &= g(X, Y) - \eta(X) \eta(Y) \\
R(X,Y)\xi &= \eta(X)Y - \eta(Y)X \\
R(\xi,X)Y &= -g(X,Y)\xi + \eta(Y)X \\
S(X,\xi) &= -(n-1)\eta(X) \\
S(\varphi X, \varphi Y) &= S(X,Y) + (n-1)\eta(X) \eta(Y) \\
\eta(R(X,Y)Z) &= g(X,Z)\eta(Y) - g(Y,Z)\eta(X)
\end{align*}
\]

Where \( R \) is the Riemannian curvature and \( S \) is the Ricci tensor of \( g \).

Let \((M, \varphi, \xi, \eta, g)\) be a nearly Kenmotsu manifold. In ([6]) it is proven that

\[
\nabla_X \xi = X - \eta(X)\xi, \quad \nabla_\xi \xi = 0.
\]

The Projective curvature tensor \( P \) ([1]) and Conharmonic curvature tensor \( N \) ([5]) on Riemannian manifold are defined respectively as

\[
\begin{align*}
P(X,Y)Z &= R(X,Y)Z - \frac{1}{n-1} [S(Y,Z)X - S(X,Z)Y] \\
N(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)r(X) - g(X,Z)r(Y)]
\end{align*}
\]

Where \( R \) is the Riemannian curvature and \( S \) is the Ricci tensor and \( r \) is the scalar curvature.

A Riemannian manifold \( M \) is said to be Einstein manifold if its Ricci tensor \( S \) of type \((0, 2)\) is of the form

\[
S(X,Y) = kg(X,Y)
\]

For all \( X, Y \in \chi(M) \) and \( k \) is certain scalar function on \( M \).
3. AN EINSTEIN NEARLY KENMOTSU MANIFOLD SATISFYING $R(X,Y).P = 0$

In this section we assume that

\[ R(X,Y).P(U,V)W = 0 \]

Let the Riemannian Manifold $M$ be an Einstein manifold, then (2.12) gives

\[ P(X,Y)Z = R(X,Y)Z - \frac{k}{n-1} [g(Y,Z)X - g(X,Z)Y] \]

Now, (3.2) can be written as

\[ \phi(X,Y,Z,W) = \phi(X,Y,Z,W) - \frac{k}{n-1} [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \]

Where $\phi(X,Y,Z,W) = g(P(X,Y)Z, W)$ and $\phi(X,Y,Z,W) = g(R(X,Y)Z, W)$

Using (2.10) in (3.3), we get

\[ \eta(P(X,Y)Z) = \left( 1 + \frac{k}{n-1} \right) [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \]

Taking $X = \xi$ in (3.4) and using (2.4) we get

\[ \eta(P(\xi,Y)Z) = \left( 1 + \frac{k}{n-1} \right) [\eta(Y)\eta(Z) - g(Y,Z)] \]

Again taking $Z = \xi$ in (3.4) and using (2.4), we get

\[ \eta(P(X,Y)\xi) = 0 \]

Now


\[ -P(U,R(X,Y)V)W - P(U,V)R(X,Y)W \]

In view of (3.1), we have

\[ R(X,Y)P(U,V)W - P(R(X,Y)U,V)W \]

\[ -P(U,R(X,Y)V)W - P(U,V)R(X,Y)W = 0 \]

Therefore,

\[ g[R(X,Y)P(U,V)W, \xi] - g[P(R(X,Y)U,V)W, \xi] \]

\[ -g[P(U,R(X,Y)V)W, \xi] - g[P(U,V)R(X,Y)W, \xi] = 0 \]

In view of (2.7) and (3.4) it follows that

\[ -\phi(U,V,W,Y) + \eta(Y)\eta(P(U,V)W) - \eta(U)\eta(P(Y,V)W) - \eta(V)\eta(P(U,Y)W) \]

\[ -\eta(W)\eta(P(U,V)Y) + g(Y,U)\eta(P(\xi,V)W) + g(Y,V)\eta(P(U,\xi)W) = 0 \]

Taking $Y = U$ in (3.8), we get

\[ -\phi(U,V,W,U) - \eta(V)\eta(P(U,U)W) - \eta(W)\eta(P(U,V)U) \]

\[ + g(U,U)\eta(P(\xi,V)W) + g(U,V)\eta(P(U,\xi)W) = 0 \]

Let $\{e_i\}, i = 1, 2, 3, \ldots, n$ be an orthonormal basis of tangent space at any point.

Then the sum for $1 \leq i \leq n$ of the relation (3.9) for $U = e_i$, gives

\[ \eta(P(\xi,Y)Z) = \frac{1}{n} S(V,W) - \left[ \frac{k}{n} + \frac{1}{n} + \frac{k}{n(n-1)} \right] g(V,W) \]

\[ + \left[ \frac{k}{n} + \frac{1}{n} + \frac{(n-1)}{n} \right] \eta(V)\eta(W) \]

Using (3.5) in (3.10) and after simplification, we get

\[ S(V,W) = -(n-1)g(V,W) \]

This gives

\[ k = -(n-1) \]

Using (3.12) in (3.8), we get

\[ -\phi(U,V,W,Y) = 0 \]
From above it follows that

\[(3. 13) \quad P(U, V)W = 0\]

Hence, we can state the following theorem:

**Theorem 1. 1.** If in an Einstein nearly Kenmotsu manifold \( M \), the relation \( R(X, Y) . P = 0 \) holds, then the manifold is projectively flat.

Next, let us suppose that the Einstein nearly Kenmosu manifold is projectively flat, that is \( P(X, Y)Z = 0 \). Then from (3. 3) we have

\[(3. 14) \quad R(X, Y, Z, W) = \frac{k}{n-1} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]\]

Where \( R(X, Y, Z, W) = g(R(X, Y)Z, W) \)

Putting \( X = W = \xi \) in (3. 14) and using (2. 4) and (2. 7), we get

\[(3. 15) \quad \left(1 + \frac{k}{n-1}\right) [\phi(Y)\phi(Z) - g(Y, Z)] = 0\]

This shows that either \( k = -(n-1) \) or \( \phi(Y)\phi(Z) = g(Y, Z) \).

But if \( g(Y, Z) = \phi(Y)\phi(Z) \), then from (2. 3) we get \( g(\varphi Y, \varphi Z) = 0 \), which is not possible.

Therefore, \( k = -(n-1) \). Now putting \( k = -(n-1) \) in (3. 2) and using \( P(X, Y)Z = 0 \), we get

\[(3. 16) \quad R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y]\]

Therefore the manifold is of constant scalar curvature \(-1\).

Hence, we can state the following theorem:

**Theorem1. 2.** A projectively flat Einstein nearly Kenmotsu manifold is locally isometric to hyperbolic space \( H^n(-1) \).

### 4. AN EINSTEIN NEARLY KENMOTSU MANIFOLD SATISFYING \( R(X, Y). N = 0 \)

In this section we assume that

\[(4. 1) \quad R(X, Y). N(U, V)W = 0\]

Let the Riemannian Manifold \( M \) be an Einstein manifold, then (2. 12) gives

\[(4. 2) \quad N(X, Y)Z = R(X, Y)Z - \frac{2k}{n-2} [g(Y, Z)X - g(X, Z)Y]\]

Now, (4. 2) can be written as

\[(4. 3) \quad \phi(N(X, Y, Z, W)) = R(X, Y, Z, W) - \frac{2k}{n-2} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]\]

Where \( \phi(N(X, Y, Z, W)) = g(N(X, Y)Z, W) \) and \( \phi(R(X, Y, Z, W)) = g(R(X, Y)Z, W) \)

Using (2. 10) in (4. 3), we get

\[(4. 4) \quad \eta(N(X, Y)Z) = \left(1 + \frac{2k}{n-2}\right) [(g(X, Z)\phi(Y) - g(Y, Z)\phi(X))]\]

Taking \( X = \xi \) in (3. 4) and using (2. 4) we get

\[(4. 5) \quad \eta(N(\xi, Y)Z) = \left(1 + \frac{2k}{n-2}\right) [(\phi(Y)\phi(Z) - g(Y, Z))]\]

Again taking \( Z = \xi \) in (3. 4) and using (2. 4), we get

\[(4. 6) \quad \eta(N(X, Y)\xi) = 0\]

Now

In view of (4.1), we have
\[ R(X,Y)N(U,V)W - N(R(X,Y)U,V)W \]
\[ - N(U,R(X,Y)V)W - N(U,V)R(X,Y)W = 0 \]
Therefore,
\[ g[R(X,Y)N(U,V)W, \xi] - g[N(R(X,Y)U,V)W, \xi] \]
\[ - g[N(U,R(X,Y)V)W, \xi] - g[N(U,V)R(X,Y)W, \xi] = 0 \]
In view of (2.7) and (4.4) it follows that
\[ -\eta(U,V) + \eta(Y)\eta(N(U,V)W) - \eta(U)\eta(N(Y,V)W) \]
\[ + \eta(V)\eta(N(U,Y)W) \]
\[ \eta(Y)\eta(N(U,V)Y) + g(Y,U)\eta(N(\xi,V)W) + g(Y,V)\eta(N(U,\xi)W) = 0 \]
Taking \( Y = U \) in (4.8), we get
\[ -\eta(U,V,W,Y) + \eta(U)\eta(N(U,V)W) - \eta(U)\eta(N(Y,V)W) \]
\[ + g(U,U)\eta(N(\xi,V)W) + g(U,V)\eta(N(U,\xi)W) = 0 \]
Let \( \{e_i\}, i = 1, 2, 3, \ldots, n \) be an orthonormal basis of tangent space at any point.
Then the sum for \( 1 \leq i \leq n \) of the relation (3.9) for \( U = e_i \), gives
\[ \eta(N(\xi,V)Z) = \frac{1}{n} S(V, W) - \frac{1}{n} \left[ 1 + \frac{2nk}{(n-2)} \right] g(V, W) \]
\[ + \frac{1}{n} \left[ \frac{2k}{(n-2)} + \frac{2(n-1)k}{n-2} \right] \eta(V)\eta(W) \]
Using (4.10) and after simplification, we get
\[ S(V, W) = -(n-1)g(V, W) - n(n-1) \left[ 1 + \frac{2k}{(n-2)} \right] \eta(V)\eta(W) \]
Putting \( W = \xi \) in above and using (2.12), yields
\[ k = -(n-2)/2 \]
Using (4.12) in (4.8), we get
\[ -\eta(U,V,W,Y) = 0 \]
From above it follows that
\[ N(U,V)W = 0 \]
Hence, we can state the following theorem:

**Theorem 1.3.** If in an Einstein nearly Kenmotsu manifold \( M \), the relation
\[ R(X,Y), N = 0 \] holds, then the manifold is projectively flat.
Next, let us suppose that Einstein nearly Kenmotsu manifold is conhormonically flat, that is \( N(X,Y)Z = 0 \). Then from (4.3) we have
\[ 'R(X,Y,Z,W) = \frac{2k}{n-2} \left[ g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \right] \]
Where \( 'R(X,Y,Z,W) = g(R(X,Y)Z,W) \)
Putting \( X = W = \xi \) in (4.14) and using (2.4) and (2.7), we get
\[ \left( 1 + \frac{2k}{n-2} \right) [\eta(Y)\eta(Z) - g(Y,Z)] = 0 \]
This shows that either \( k = -(n-2)/2 \) or \( \eta(Y)\eta(Z) = g(Y,Z) \).
But if \( g(Y,Z) = \eta(Y)\eta(Z) \), then from (2.3) we get \( g(\varphi Y, \varphi Z) = 0 \), which is not possible.
Therefore, \( k = -(n-2)/2 \). Now putting \( k = -(n-2)/2 \) in (4.2) and using \( N(X,Y)Z = 0 \), we get
Therefore the manifold is of constant scalar curvature $-1$.

Hence, we can state the following theorem:

**Theorem 1.4.** A conharmonically flat Einstein nearly Kenmotsu manifold is locally isometric to hyperbolic space $H^n(-1)$.

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