Fuzzy semi-regular subset of Fuzzy topological space

Md. Arshaduzzaman
Department of Mathematics,
G.B. College, Naugachia, Bhagalpur, Bihar (India).

Abstract
The intent of this paper is to study about some subspaces of a fuzzy topological space i.e. fuzzy semi-closed and fuzzy semi-regular sub-space, externally disconnected sub-space. We also obtain some properties of such spaces relative to the fuzzy topological space.

INTRODUCTION:
L.A. Zadeh\(^1\) was the first Mathematician who invented fuzzy set and placed before us very interesting characteristics\(^1\).

A fuzzy set \(\mathcal{A}\) is an universal set \(X\) is a mapping \(\mathcal{A}: X \rightarrow [0,1]\). The null fuzzy set \(O\) is a mapping from \(X\) to \([0,1]\) which admits of the value \(O\) and the fuzzy set \(1\) is mapping from \(X\) to \([0,1]\) which admits of value \(1\) only.

A fuzzy set \(\lambda\) on \(X\) is called a fuzzy singleton if it takes the value \(O\) for a except one.

The point at which a fuzzy singleton takes the non zero value is called the support of the singleton \(2\).

A family \(\mathcal{I}\), where \(I = [0, 1]\) is called a fuzzy topology for \(X\) if

\[
\mathcal{I}_1: \forall \mathcal{A} \in \mathcal{I}, \mathcal{B} \in \mathcal{I}
\]

\[
\mathcal{I}_2: \forall \lambda, \mu \in \mathcal{I}, \lambda \wedge \mu \in \mathcal{I}
\]

\[
\mathcal{I}_3: \forall \lambda \in \mathcal{I}, \lambda \wedge j \in \mathcal{I}, \lambda \wedge j \in \mathcal{I} (j=1, 2, 3, \ldots)
\]
The pair  is called a fuzzy topological space\(^3\).

The members of    are called -fuzzy open sets. A fuzzy set \( U \) is called \( \mathcal{I} \)-fuzzy closed if its complement \( U \in \mathcal{I} \).

The closure \( \text{Int}(\lambda) \) and the interior

\[
\text{Int}(\lambda) = \{ U : U \text{ is a fuzzy open subset of } \lambda \}
\]

A fuzzy subset \( \lambda \) of \( X \) is called fuzzy semi open if a fuzzy open set \( \tilde{U} \) of \( X \) such that

\[
\tilde{U} \subseteq \lambda \subseteq \text{cl}(\tilde{U})
\]

Their \( \lambda \) is called fuzzy semi closed \(^4,5\).

The semi closure \( \text{Int}(\lambda) \) and the semi interior \( \text{Int}(\lambda) \) of a fuzzy set \( \lambda \) are defined\(^6\) by

\[
\text{cl}(\lambda) = \{ K : K \text{ is a semiclosed superset of } \lambda \}
\]

\[
\text{Int}(\lambda) = \{ K : K \text{ is a semiopen subset of } \lambda \}
\]

A sub set of \( X \) is called fuzzy semi regular, if it is both fuzzy semi-open and fuzzy semi closed\(^7\).

2. FUZZY SEMI CLOSED AND SEMI REGULAR SUB SPACE :

Definition (2.1):
A topological space \((X, \mathcal{I})\) is said to be fuzzy semi closed if corresponding to every
cover \( C = \{ \lambda \alpha : \alpha \in \Delta \} \) by fuzzy semi open subsets of \( X \), \( \exists \) finite fuzzy subset \( \lambda \alpha_0 \) such that
\[
X = \bigvee \left\{ \text{cl} \left( \lambda \alpha_0 \right)_S \right\} : \alpha_0 \in \Delta
\]

A fuzzy subset \( \lambda \) of \( X \) is called fuzzy semi closed relative to \( (X, \mathcal{I}) \) if for every cover \( C = \{ \lambda \alpha : \alpha \in \Delta \} \) by fuzzy semi open subsets of \( X \), \( \exists \) finite fuzzy subset \( \lambda \alpha_0 \) such that
\[
X = \bigvee \left\{ \text{cl} \left( \lambda \alpha_0 \right)_S \right\} : \alpha_0 \in \Delta
\]

A fuzzy subset \( \lambda \) of \( X \) is called fuzzy semi closed relative to \( (X, \mathcal{I}) \) if for every cover \( C = \{ \lambda \alpha : \alpha \in \Delta \} \) of by fuzzy semi open sets of \( X \), \( \exists \) a finite subset \( \lambda \alpha_0 \) such that
\[
\lambda \leq \bigvee \left\{ \text{cl} \left( \lambda \alpha_0 \right)_S \right\} : \alpha_0 \in \Delta
\]

**Definition (2.2):**

A fuzzy topological space \( (X, \mathcal{I}) \) is said to be fuzzy semi-regular if for each fuzzy closed set \( U \) and a fuzzy point \( a \), a pair of disjoint fuzzy semi open sets in such that a pair of disjoint fuzzy semi open sets in \( X \) such that.

**Theorem (2.3):**

A topological space \( (X, \mathcal{I}) \) is fuzzy semi-closed if every proper fuzzy semi-regular subset of \( X \) is fuzzy semi-closed relative to \( (X, \mathcal{I}) \).

**Proof:**

Let \( p \) be a proper fuzzy semi-regular subset of \( X \). Let \( \{ \lambda \alpha : \alpha \in \Delta \} = C \) be a fuzzy cover \( p \) such that \( \lambda \alpha \) is a member of fuzzy semi open subsets of \( X \) for each. Then \( X-p \) is also fuzzy semi regular \( \Rightarrow C \cup \left\{ 1-p \right\} \) forms a cover of.

Since \( X \) is fuzzy semi closed, \( \exists \) a finite sub family such that,
\[ X = \vee \left\{ \text{cl}\left( \lambda \alpha_0 \right) \right\} \] 
\[ \Rightarrow p \leq \vee \left\{ \text{cl}\left( \lambda \alpha_0 \right), \alpha_0 \in \Delta \right\} \]

Conversely let \( \mathcal{C} = \left\{ \lambda \alpha : \alpha \in \Delta \right\} \) be a cover of \( X \), where \( \alpha \) is a member of fuzzy semi open set \( \forall \alpha \in \Delta \)

\[ p = \vee \left\{ \text{cl}\left( \lambda \alpha_0 \right) \right\} \text{ forsome } \alpha_0 \in \Delta \]

Since \( p \) is a member of fuzzy semi-regular subset of \( X \), so is 1-\( p \) and 1-\( p \leq \vee \{ \lambda \alpha : \alpha \in \Delta \} \).

Since 1-\( p \) is fuzzy semi closed relative to \( X \), \( \exists \) a finite subset such that

\[ X - p \leq \vee \left\{ \text{cl}\left( \lambda \alpha_0 \right), \alpha_0 \in \Delta \right\} \]

\[ \Rightarrow X = \vee \left\{ \left[ \text{cl}(\lambda \alpha) \right]_{S}, \alpha \in \Delta \cup \lambda \alpha_0 \right\} \]

\[ \Rightarrow X \text{ is fuzzy semi closed.} \]

**Remarks (2.4):**

For a fuzzy subset \( \lambda \) of a space \( X \), the following conditions are equivalent.

(i) \( \lambda \) is semi closed relative to \( X \).

(ii) Every cover of \( \lambda \) by fuzzy semi open subsets of \( X \) has a finite sub cover.

(iii) Every cover of \( \lambda \) by fuzzy semi regular subsets of \( X \) has a finite sub cover.

**Theorem (2.5):**

Let \( \tilde{\lambda} \) and \( \tilde{\mu} \) be two fuzzy subsets of a space \( X \) such that \( \tilde{\lambda} \leq \tilde{\mu} \leq X \), where \( \tilde{\mu} \) is a fuzzy semi open subset, then if \( \tilde{\lambda} \) be fuzzy semi closed relative to \( X \), it is fuzzy semi closed relative to \( \tilde{\mu} \) also.
**Proof:**

Let \( X = \left\{ \lambda \alpha: \alpha \in \Delta \right\} \) be a cover of \( \lambda \) and \( \lambda \alpha \) be a fuzzy semi open subset of \( \mu \) for all \( \alpha \in \Delta \). Since \( X \) is a fuzzy open subset of \( X \), so is \( \lambda \alpha \). Since \( \lambda \) is fuzzy semi closed relative to \( X \), a finite sub family \( \{\alpha_0\} \) such that

\[
\lambda \leq \bigvee \{ \text{cl}(\lambda \alpha_0) \}_X, \alpha_0 \in \Delta \}
\]

\[
\Rightarrow \lambda \leq \bigvee \{ \text{cl}(\lambda \alpha_0) \}_X \cap \mu
\]

Hence

\[
\lambda \leq \bigvee \{ \text{cl}(\lambda \alpha_0) \}_\mu, \alpha_0 \in \Delta \}
\]

\[
\Rightarrow \lambda \text{ is fuzzy semi closed relative to sub space } \mu.
\]

**3. DISCONNECTED AND SEMI HAUSDORFF SPACE:**

**Definition (3.1):**

A fuzzy topological space is said to be extremely disconnected if \( \text{cl}(\hat{U}) \) is fuzzy open in \( X \) for every fuzzy\(^5\) open set \( \hat{U} \) of \( X \).

**Remarks (3.2):**

If \( X \) is an extremely disconnected fuzzy topological space and \( \lambda \) is fuzzy semi regular subset of \( X \), then \( \lambda \) is fuzzy closed and fuzzy open in \( X \).

**Remarks (3.3):**

A fuzzy open set of a space \( X \) is fuzzy semi-closed as sub space of \( X \), iff it is fuzzy semi closed relative to \( X \).

**Theorem (3.4):**

An extremely disconnected fuzzy topological space \( X \) is fuzzy semi closed if every fuzzy semi-regular subset of \( X \) is a fuzzy semi closed sub-space of \( X \).
Proof:

Let \( \mathcal{C} = \{ \lambda \alpha : \alpha \Delta \} \) be a fuzzy cover of \( X \), where \( \lambda \alpha \) is a fuzzy semi open subset of \( X \) for all \( \alpha \in \Delta \).

Suppose that \( 1 \neq \left[ \mathrm{cl}_X \left( \lambda \beta \right) \right]_S \neq 0 \)

Since \( \lambda \beta \) is a fuzzy semi open subset of \( X \), so \( \left[ \mathrm{cl}_X \left( \lambda \beta \right) \right]_S \) is a fuzzy semi regular subset of \( X \).

\[
\Rightarrow 1 - \left[ \mathrm{cl}_X \left( \lambda \beta \right) \right]_S \text{ is a fuzzy semi regular subset of } X.
\]

\[
\Rightarrow 1 - \left[ \mathrm{cl}_X \left( \lambda \beta \right) \right]_S \text{ is a fuzzy semi closed subset and hence both fuzzy semi open and}
\]

semi closed in \( X \), by remark (3.2), so \( 1 - \left[ \mathrm{cl}_X \left( \lambda \beta \right) \right]_S \) is fuzzy semi closed relative to \( X \), by remarks (3.3),

\[
\Rightarrow 1 - \left[ \mathrm{cl}_X \left( \lambda \beta \right) \right]_S \leq \bigvee \left\{ \lambda \alpha : \alpha \in \Delta \right\}, \exists
\]

a finite sub family \( \lambda \alpha_0 \) such that,

\[
X = \bigvee \left\{ \left[ \mathrm{cl}_X \left( \lambda \alpha \right) \right]_S \cup \lambda \beta \right\}
\]

\[
\Rightarrow X \text{ is fuzzy semi closed.}
\]

REFERENCES


