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Development of Alternative Displacement Potential Formulation of Isotropic Materials

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Abstract

A new mathematical displacement potential function formulation has been developed. Step by step calculations are shown for obtaining the formulations. Firstly the two displacement components are assumed in terms of the functions .The two displacement components are substituted in the 2nd equlibrium equations and the constants are found and these constants are applied in the 1st equlibrium equations and the 1st equilibrium equation turns into the governing equation which is necessary for solving the problem. By using the unknown constants the required stress and displacement boundary conditions are obtained. The technique of analytical solution of a stiffened bar under unspecified boundary conditions are evaluated by using the present developed method.

Keywords: Alternative Displacement Potential Formulation, Isotropic material, Stiffened bars.

Introduction
Elasticity problems are usually formulated either in terms of deformation parameters or stress parameters. Among the existing mathematical models of plane boundary-value stress problems, the stress function approach. [1] and the displacement formulations [2] are noticeable The displacement formulation, on the other hand, involves finding two displacement components simultaneously from the two second-order partial differential equations of equilibrium, which is extremely difficult and this problem becomes more serious when the boundary conditions are mixed [2]. Successful application of the displacement potential formulation in conjunction with Fourier integral and finite-difference technique have been reported for the solution of plane elastic problems where all the conditions on the boundary are mixed [3, 4].

Hossain Md. Zubaer [5] developed a displacement potential formulation of 3D isotropic structures and he solved a beam problem by using the formulation. For proving the soundness of the formulation, he compared the result obtained from the displacement formulation with the solution of FEM.. Recently Nath and Ahmed *et al.* .[6-7] solved tire tread problem in conjunction with FDM technique. Here is shown how a new alternative displacement formulation has been developed for solving 2 D plain stress/ Plain strain problems of isotropic materials.

New Displacement Potential Formulation For Isotropic Material

With reference to a rectangular coordinate system, in absence of body forces, these three variables are governed by the following two equilibrium equations [1].

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \sigma_{yy}}{\partial \mathrm{y}} + \frac{\partial \sigma_{xy}}{\partial \mathrm{x}} = 0 \tag{2}$$

In the displacement potential function formulation, the displacement components are expressed in terms of a potential function ψ of space variables as follows:

$$u_x = \alpha_1 \frac{\partial^2 \psi}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_3 \frac{\partial^2 \psi}{\partial y^2} \tag{3}$$

$$u_y = \alpha_4 \frac{\partial^2 \psi}{\partial x^2} + \alpha_5 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_6 \frac{\partial^2 \psi}{\partial y^2} \tag{4}$$

Here, α's are unknown material constants.
The stress and stain relationship of a two-dimensional orthotropic composite structures are given below in a matrix form.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{5}$$

where, $Q_{11} = \frac{E}{1-\mu^2}; Q_{12} = \frac{\mu E}{1-\mu^2}$

$$Q_{22} = \frac{E}{1-\mu^2}\ ;\ Q_{66} = \frac{E}{2(1+\mu)}$$

From the equation (5), we get the following equations.

$$\sigma_{xx} = Q_{11}\varepsilon_x + Q_{12}\varepsilon_y = Q_{11}\frac{\partial u_x}{\partial x} + Q_{12}\frac{\partial u_y}{\partial y} \tag{6}$$

$$\sigma_{yy} = Q_{12}\varepsilon_x + Q_{22}\varepsilon_y = Q_{12}\frac{\partial u_x}{\partial x} + Q_{22}\frac{\partial u_y}{\partial y} \tag{7}$$

$$\sigma_{xy} = Q_{66}\gamma_{xy} = G\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] \tag{8}$$

Combining equations [3-4] and [6-8], we get the three following three stress components in terms of the displacement potential function ψ.

$$\sigma_{xx} = \alpha_1 Q_{11}\frac{\partial^3\psi}{\partial x^3} + (\alpha_2 Q_{11} + \alpha_4 Q_{12})\frac{\partial^3\psi}{\partial x^2\partial y} + (\alpha_3 Q_{11} + \alpha_5 Q_{12})\frac{\partial^3\psi}{\partial x\partial y^3} + \alpha_6 Q_{12}\frac{\partial^3\psi}{\partial y^3} \tag{9}$$

$$\sigma_{yy} = \alpha_1 Q_{12}\frac{\partial^3\psi}{\partial x^3} + (\alpha_2 Q_{12} + \alpha_4 Q_{22})\frac{\partial^3\psi}{\partial x^2\partial y} + (\alpha_3 Q_{12} + \alpha_5 Q_{22})\frac{\partial^3\psi}{\partial x\partial y^3} + \alpha_6 Q_{22}\frac{\partial^3\psi}{\partial y^3} \tag{10}$$

$$\sigma_{xy} = \alpha_4 Q_{66}\frac{\partial^3\psi}{\partial x^3} + (\alpha_1 Q_{66} + \alpha_5 Q_{66})\frac{\partial^3\psi}{\partial x^2\partial y} + (\alpha_2 Q_{66} + \alpha_6 Q_{66})\frac{\partial^3\psi}{\partial x\partial y^3} + \alpha_3 Q_{66}\frac{\partial^3\psi}{\partial y^3} \tag{11}$$

Combining equations [1-2] and [9-11], we obtain the equilibrium equations in terms of the function ψ (x,y), which are

$$\alpha_1 Q_{11}\frac{\partial^4\psi}{\partial x^4} + \{\alpha_2 Q_{11} + \alpha_4(Q_{12} + Q_{66})\}\frac{\partial^4\psi}{\partial x^3\partial y} + \{\alpha_3 Q_{11} + \alpha_1 Q_{66} + \alpha_5(Q_{12} + Q_{66})\}\frac{\partial^4\psi}{\partial x^2\partial y^2} + \{\alpha_2 Q_{66} + \alpha_6(Q_{12} + Q_{66})\}\frac{\partial^4\psi}{\partial x\,\partial y^3} + \alpha_3 Q_{66}\frac{\partial^4\psi}{\partial y^4} = 0 \tag{12}$$

$$\alpha_4 Q_{66}\frac{\partial^4\psi}{\partial x^4} + \{\alpha_5 Q_{66} + \alpha_1(Q_{12} + Q_{66})\}\frac{\partial^4\psi}{\partial x^3\partial y} + \{\alpha_4 Q_{22} + \alpha_6 Q_{66} + \alpha_2(Q_{12} + Q_{66})\}\frac{\partial^4\psi}{\partial x^2\partial y^2} + \{\alpha_5 Q_{22} + \alpha_3(Q_{12} + Q_{66})\}\frac{\partial^4\psi}{\partial x\,\partial y^3} + \alpha_6 Q_{22}\frac{\partial^4\psi}{\partial y^4} = 0 \tag{13}$$

The constants, α's are chosen here in such a way that 2nd equation of Eqs.(13) is automatically satisfied under all circumstances. This will happen when coefficients of all derivatives present in 2nd equation of Eqs (13) are individually zero. That is, when

$$\alpha_4 Q_{66} = 0 \tag{14}$$

$$\{\alpha_5 Q_{66} + \alpha_1 (Q_{12} + Q_{66})\} = 0 \tag{15}$$

$$\{\alpha_4 Q_{22} + \alpha_6 Q_{66} + \alpha_2 (Q_{12} + Q_{66})\} = 0 \tag{16}$$

$$\{\alpha_5 Q_{22} + \alpha_3 (Q_{12} + Q_{66})\} = 0 \tag{17}$$

$$\alpha_6 Q_{22} = 0 \tag{18}$$

From equations [14,17,18], we obtain

$$\alpha_2 = \alpha_4 = \alpha_6 = 0$$

$$\alpha_5 = -\frac{\alpha_1 (Q_{12} + Q_{66})}{Q_{66}}$$

$$\alpha_3 = -\frac{\alpha_5 Q_{22}}{(Q_{12} + Q_{66})}$$

Thus for ψ to be a solution of the stress problem, it has to satisfy Eqs.(12) only. However, the values of α's must be known in advance. Here, we have basically two equations [12-13] for determining three unknown α's. An arbitrary value is thus assigned to any one of these three unknowns and the remaining α's are solved from Equation (13) Assuming α_1=1, the values of α's thus obtained, are as follows

$$\alpha_2 = \alpha_4 = \alpha_6 = 0$$

$$\alpha_1 = 1$$

$$\alpha_5 = -\frac{1+\mu}{1-\mu}$$

$$\alpha_3 = \frac{2}{(1-\mu)}$$

When the above values of α's are situated in the first equation of Eqs.(12), the governing differential equation for the solution of two dimensional isotropic structure problems becomes

$$\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = 0 \tag{19}$$

By putting the α's value in equations [1-2, 9-11], we obtain the boundary conditions of different displacement and stress boundary conditions for solving the problem.

Table 1: The two formulations including boundary conditions and governing equations

Alternative New Displacement Potential Formulation	Existing Displacement Potential Formulation
$u_x(x,y) = \frac{\partial^2\psi}{\partial x^2} + \frac{2}{(1-\mu)}\frac{\partial^2\psi}{\partial y^2}$	$u_x(x,y) = \frac{\partial^2\psi}{\partial x\partial y}$
$u_y(x,y) = -\frac{1+\mu}{1-\mu}\frac{\partial^2\psi}{\partial x\partial y}$	$u_y(x,y) = -\frac{1}{1+\mu}\left[2\frac{\partial^2\psi}{\partial x^2} + (1-\mu)\frac{\partial^2\psi}{\partial y^2}\right.$
$\sigma_{xx}(x,y) = \frac{E}{1-\mu^2}\left[\frac{\partial^3\psi}{\partial x^3} + (2+\mu)\right.$	$\sigma_{xx}(x,y) = \frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^2\partial y} - \mu\frac{\partial^3\psi}{\partial y^3}\right]$
$\sigma_{yy}(x,y) = \frac{E}{1-\mu^2}\left[\mu\frac{\partial^3\psi}{\partial x^3} - \mu\right.$	$\sigma_{yy}(x,y) = -\frac{E}{(1+\mu)^2}\left[(2+\mu)\frac{\partial^3\psi}{\partial x^2\partial y} + \right.$
$\sigma_{xy}(x,y) = \frac{E}{1-\mu^2}\left[-\mu\frac{\partial^3\psi}{\partial x^2\partial y} + \right.$	$\sigma_{xy}(x,y) = -\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^3} - \mu\right.$
$\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = 0$	$\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = 0$

$$u_x(x,y) = \frac{\partial^2\psi}{\partial x^2} + \frac{2}{(1-\mu)}\frac{\partial^2\psi}{\partial y^2} \tag{20}$$

$$u_y(x,y) = -\frac{1+\mu}{1-\mu}\frac{\partial^2\psi}{\partial x\partial y} \tag{21}$$

$$\sigma_{xx}(x,y) = \frac{E}{1-\mu^2}\left[\frac{\partial^3\psi}{\partial x^3} + (2+\mu)\frac{\partial^3\psi}{\partial x\partial y^2}\right] \tag{22}$$

$$\sigma_{yy}(x,y) = \frac{E}{1-\mu^2}\left[\mu\frac{\partial^3\psi}{\partial x^3} - \mu\frac{\partial^3\psi}{\partial x\partial y^2}\right] \tag{23}$$

$$\sigma_{xy}(x,y) = \frac{E}{1-\mu^2}\left[-\mu\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3}\right] \tag{24}$$

Solution Technique of The Problem

For the model shown in Fig.1, the stiffened bar is considered to be of unit thickness and the potential function ψ is assumed to be

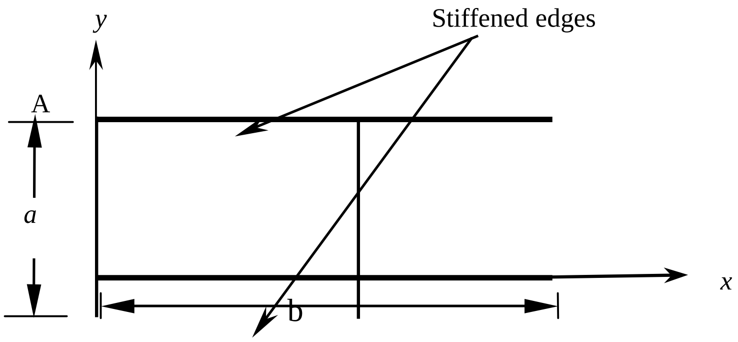

Figure 1: Analytical model of the problem.

$$\psi = \sum_{m=1}^{\infty} X_m \sin\alpha y \tag{25}$$

where X_m is a function of x only and $\alpha = m\pi/a$. Substituting Eq.[25] in Eq.[19], we get

$$X_m'''' - 2\alpha^2 X_m'' + \alpha^4 X_m = 0 \tag{26}$$

where the ($'$) indicates differentiation with respect to x. The general solution of this differential equation can be given by:

$$X_m = A_m\cosh\alpha x + B_m\alpha x\sinh\alpha x + C_m\sinh\alpha x + D_m\alpha x\cosh\alpha x \tag{27}$$

Here A_m, B_m, C_m, and D_m are constants. Now combining Eqs.(20), (21), (22), (23), (24), (25), and (27) the expressions of stress and displacement components are obtained as follows:

$$u_x(x,y) = \sum_{m=1}^{\infty}\left[-X_m\alpha^2 + \frac{2}{1-\mu}X_m''\right]\alpha X_m' \, sin\alpha y \tag{28}$$

$$u_y(x,y) = -\sum_{m=1}^{\infty}\left[\frac{1+\mu}{1-\mu}\alpha X_m'\right]\cos\alpha y \tag{29}$$

$$\sigma_{xx}(x,y) = \frac{E}{1-\mu^2}\sum_{m=1}^{\infty}\left[X_m''' - (2+\mu)\alpha^2 X_m'\right]\sin\alpha y \tag{30}$$

$$\sigma_{yy}(x,y)=\frac{E}{1-\mu^2}\sum_{m=1}^{\infty}\left[\mu X_m'''+\alpha^2 X_m'\right]\sin\alpha y \quad (31)$$

$$\sigma_{xy}(x,y)=\frac{E}{1-\mu^2}\sum_{m=1}^{\infty}\left[\mu\alpha^3 X_m-\alpha^3 X_m\right]cos\alpha y \quad (32)$$

For the present problem, it is seen that the boundary conditions on stiffened edges $u_x = 0 \quad at \quad y = 0 \; and \; y = a \,; \sigma_{yy} = 0 \quad at \quad y = 0 \; and \; y = a$ are satisfied automatically. By applying relevant boundary conditions on other two boundaries of the bar, we can solve the problem by using the boundary expressions[28-32]

Conclusions

An alternative method formulation has been developed for solving any isotropic structures of having mixed mode of boundary conditions. The boundary conditions and governing equations are expressed in terms of the displacement potential function, ψ .The boundary conditions and governing equations having the present developed method and the previous method are shown in tabular form. The boundary conditions both of the methods are quietly different but the governing equations for both of the methods are same. The present method satisfies the 2nd equilibrium equation and the governing equation comes from the first equilibrium equation but the previous displacement formation satisfies the 1st equilibrium equation and the governing equation comes from the 2nd equation. The present method should be more reliable because all of the boundary equations of the present method exists material property but on the other hand all of the boundary conditions of the previous displacement formulation does not exist material property. The technique of the solving of a stiffened bar under unspecified boundary conditions like loadings or constrains are shown analytically. The present formulation will be a reliable guideline for solving any isotropic structures because all of the boundary conditions satisfy equilibrium equations.

Nomenclature

ψ Displacement Potential function

E Modulus of Elasticity

μ Poisson's ratio

u_x, u_y Displacement component in x and y directions

$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ Normal stress component *x* and *y* diretions and shear stress component.

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