International Journal of Statistics and Systems
ISSN 0973-2675 Volume 19, Number 2 (2024), pp. 29-43

https://dx.doi.org/10.37622/IJSS/19.1.2024.13-21

A Birth-Death-Immigration-Emigration (BDIE) Process with Genocide Incidence

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Abstract

The probability generating function (PGF) for a birth- death-immigration-emigration (BDIE) Process with genocide incidence as a partial catastrophe is developed. In this paper, a BDIE process is considered under the influence of genocide incidence which, when it occurs, reduces the population size to a non-zero state. Upon utilizing the generating function, the population mean and its variance for BDIE processes with genocide can be found quite easily.

Key Words: BDIE process, PGF, Genocide, Partial Catastrophe

1. Introduction

The stochastic process on birth-death type processes have been developed mostly Yule, Feller, Kandal and Getz among others. Recently, Granita [1] modeled linear growth birth and death processes with immigration and emigration using the stochastic differential equation. Getz [2], modeled birth-death processes with positive and negative controls using the probability generating function. The construction of the transient probabilities for a simple birth-death-immigration process under the influence of total catastrophe was constructed by Randall [3] and in his paper a total catastrophe that wipes out the total population to size zero sate was taken into consideration. Di Crescenzo [4] also worked on birth-death process subject to catastrophes using the Laplace transform of its probability density function to obtain the mean and variance. Moreover, Michael [5] modeled the immigration-emigration with catastrophe and found the steady-state solution using the classic recursive methods. However, the generating function model of birth-death-immigration-emigration process with genocide incidence as a partial catastrophe that depends on population size has not been discussed in the previous works. In this paper, we shall consider a genocide which occurs at a constant rate and when it occurs, reduces the population to a non- zero state. The purpose of this paper is to analyze the BDIE processes by introducing a genocide parameter depending on the size of the population like birth and death because genocide is considered as a forced death. This led to the development of differential difference equation and obtain the generating function for BDIE with genocide incidence by using the probability generating function method which will further lead to the determination of the mean and variance of BDIE with genocide incidence.

2. Development of The Model

Let n be the number of members in a population which is a stochastic variable taking on the values 0,1,2, while $P_n(t)$ denote the probability that the population is of size n at time t. Also let $\Delta t > 0$ denote a small increment in time. If the population is of size n at time t, then during the interval $[t, t+\Delta t]$, it is assumed that any of the following eight events may occur:

(1.) Each individual present at time t may give birth to an additional individual with probability

$$\lambda(t)\Delta t + o(\Delta t)$$

(2.) An individual may be added to the population (immigration)with probability

$$\nu(t)\Delta t + o(\Delta t)$$

(3.) Each individual present at time t may die and leave the population with the probability

$$\mu(t)\Delta t + o(\Delta t)$$

(4.) Also an individual may be removed (culling, emigration) from the population with probability

$$\alpha(t)\Delta t + o(\Delta t)$$

(5.) A genocide incidence occurs in a population of size n and reduce the population size to certain lower level η with a rate γ.Therefore, the presence of genocide changes the population size from a state n to a state η with probability

$$\gamma(t)\Delta t + o(\Delta t)$$

(6.) None of the events (i)-(v) occur with probability

$$1-[\lambda+\mu+\nu+\alpha+\gamma](t)\Delta t + o(\Delta t),$$

then the population level remains the same.

(7.) We also assume that under these conditions twins, triplets, etc occurrence is $o(\Delta t)$

(8.) Lastly, we assume that the probability of more than one event (i)-(v) occurring is $o(\Delta t)$

The parameters $\lambda(t)$, $\mu(t)$, $\nu(t)$, $\alpha(t)$ and $\gamma(t)$ represent birth, death, immigration, emigration and genocide rates respectively at time t.

As in the simple birth-death process, births and deaths occur proportional to the population size with a birth rate $\lambda > 0$ and a death rate $\mu > 0$. Immigration and emigration will occur independent of the population size with rates $\nu > 0$ and $\alpha > 0$ respectively. Further, the occurrence of genocide is also dependent of population size and will occur at a rate $\gamma > 0$. Thus, using the idea of Getz, the above events can be summarized in the following transitional rates:

Table 1: BDIE Transitional Rates with Genocide

Transition	**at time Rate**	
$n \to n+1$	$\lambda n + \nu$,	$n \geq 0$
$n \to n-1$	$\mu n + \alpha$,	$n \geq 1$
$\eta \to n$	$\eta\gamma$,	$n \geq 1$

Where $\lambda := \lambda(t)$, $\mu := \mu(t)$, $\nu := \nu(t)$, $\alpha := \alpha(t)$ and $\gamma := \gamma(t)$. Table 1 is the summary of the transitional rates for the BDIE process with genocide incidence. Under these conditions, we have the probability of transition for BDIE with genocide state in the time interval $[t, t+\Delta t]$.

$$\begin{aligned} P_n(t+\Delta t) = &\{(n+1)\mu(t)\Delta t + \alpha(t)\Delta t\}P_{n+1}(t) + \{(n-1)\lambda(t)\Delta t + \nu(t)\Delta t\}P_{n-1}(t) \\ &+ \eta\{\gamma(t)\Delta t\}P\eta + \{1 - [n(\mu(t) + \lambda(t) + \gamma(t))\Delta t] - [\alpha(t)\Delta t + \nu(t)\Delta t]\}P_n(t) \end{aligned}$$

(2.1)

The objective is to compute the generating function $G(s,t)$, first we consider the transition probability from state η to n in (v) above and we have;

$$\begin{aligned} \eta\{\gamma(t)\Delta t\}P_\eta(t) + \{1 - n(\gamma(t)\Delta t)\}P_n(t) &= (\eta P_\eta(t) - nP_n(t))\gamma(t)\Delta t + P_n(t) \\ &= n\left(\frac{\eta}{n}P\eta - P_n\right)\gamma(t)\Delta t + P_n(t) \end{aligned}$$

(2.1a)

(2.1a) for large n in Equation (2.1a), $\frac{\eta}{n} \to 0$ and thus Equation(2.1a) can be approximately

$$\left[1 - n\gamma(t)\Delta t\right]P_n(t)$$

Hence Equation (2.1) becomes;

$$\begin{aligned} P_n(t+\Delta t) = &\{(n+1)\mu(t)\Delta t + \alpha(t)\Delta t\}P_{n+1}(t) + \{(n-1)\lambda(t)\Delta t + \nu(t)\Delta t\}P_{n-1}(t) \\ &+ \{1 - \left[n(\mu(t) + \lambda(t) + \gamma(t))\Delta t\right] - \left[\alpha(t)\Delta t + \nu(t)\Delta t\right]\}P_n(t) \end{aligned}$$

(2.1b)

Which upon simplifying and dropping the dependence of the parameters on time t and also letting $\Delta t \to 0$, we get:

$$\begin{aligned} P_n'(t) = &\left[\mu(n+1) + \alpha\right]P_{n+1}(t) + \left[\lambda(n-1) + \nu\right]P_{n-1}(t) \\ &- \left[n(\lambda + \mu + \gamma) + \nu + \alpha\right]P_n(t) \quad n = 1, 2, 3, \ldots \end{aligned}$$

(2.2)

Where, the prime indicates the derivative with respect to time t. For $n = 0$, we have

$$P_0'(t) = (\mu + \alpha)P_1(t) - (\nu + \alpha)P_0(t)$$

(2.3)

as a boundary condition.

3. Probability Generating Function

Here, we derive the (PGF) of Equation (2.2). For that purpose, we apply the definition PGF. By definition we let

$$G(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n$$

(3.1)

Using Equation (2.2) we obtain the PGF as

$$\begin{aligned} G'(s,t) = \sum_{n=0}^{\infty} P_n'(t)s^n = &\underbrace{\sum_{n=0}^{\infty}\left[\mu(n+1) + \alpha\right]P_{n+1}(t)s^n}_{=A} + \underbrace{\sum_{n=0}^{\infty}\left[\lambda(n-1) + \nu\right]P_{n-1}(t)s^n}_{=B} \\ &- \underbrace{\sum_{n=0}^{\infty}\left[n(\lambda + \mu + \gamma) + \nu + \alpha\right]P_n(t)s^n}_{=C} \end{aligned}$$

(3.2)

For simplicity, we solve the right hand side of Equation (3.2) and we have

$$A = \mu \frac{\partial G(s,t)}{\partial s} + \frac{\alpha}{s}\left(G(s,t) - P_0\right)$$

(3.3)

$$B = s\nu G(s,t) + s^2 \lambda \frac{\partial G(s,t)}{\partial s}$$

(3.4)

and

$$C = -(\nu + \alpha) G(s,t) - (\mu + \lambda + \gamma) s \frac{\partial G(s,t)}{\partial s}$$

(3.5)

Using Equation (3.3), (3.4) and (3.5) in Equation (3.2) we get;

$$\frac{\partial G(s,t)}{\partial s} = \left[(\lambda s - \mu)(s-1) - \gamma s\right] \frac{\partial G(s,t)}{\partial s} + \left[(s-1)\left(\nu - \frac{\alpha}{s}\right)\right] G(s,t) - \frac{\alpha}{s} P_0(t)$$

(3.6)

At time $t = 0$ the population has a distribution

$$P_n(0) = \Gamma_n, \quad n = 0,1,2,...$$

$$\sum_{n=0}^{\infty} \Gamma_n = 1$$

$$P_n(0) = \Gamma_n = \frac{1}{n}$$

(3.7a)

Due to the fact that $P_n(0)$ is a distribution,

$$Mean = n_o$$

$$Variance = \sigma^2{}_0$$

(3.7b)

Assuming $P_n(0)$ is normally distributed for all n and no emigration expected at $n = 0$, hence the term $\frac{\alpha}{s} P_0(t)$ in Equation (3.6) can be neglected.

Hence Equation (3.6) becomes

$$\frac{\partial G(s,t)}{\partial s}=\left[(\lambda s-\mu)(s-1)-\gamma s\right]\frac{\partial G(s,t)}{\partial s}+\left[(s-1)\left(\nu-\frac{\alpha}{s}\right)\right]G(s,t)$$

(3.8)

We now solve Equation (3.8), assuming λ , μ , α , ν and γ are constants(do not depend on t).The auxiliary equation to Equation(3.8)is

$$\frac{ds}{\left(\lambda s^2-(\lambda+\mu)s-\gamma s+\mu\right)}=\frac{dt}{-1}=\frac{dG(.)}{\left(\frac{\alpha}{s}-\nu\right)(s-1)G(.)}$$

(3.9)

Considering the equality of the first part of Equation (3.9, we have

$$\frac{ds}{dt}=\left(\lambda s^2-(\lambda+\mu)s-\gamma s+\mu\right)$$

(3.10)

By separation of variables Equation (3.10) becomes,

$$\int\frac{ds}{\left(s^2-\frac{(\lambda+\mu+\gamma)}{\lambda}s+\frac{\mu}{\lambda}\right)}=-\lambda t+K$$

(3.11)

Where, K is a constant of integration.

To solve Equation (3.11) we use the method of partial fractions

$$\frac{1}{s^2-\frac{(\lambda+\mu+\gamma)}{\lambda}s+\frac{\mu}{\lambda}}=\frac{1}{\left(s-(a-b)\right)\left(s-(a+b)\right)}$$

(3.12)

Where

$$a=\left(\frac{\lambda+\mu+\gamma}{2\lambda}\right)$$

(3.13)

$$b=\frac{\sqrt{\frac{(\lambda+\mu+\gamma)^2}{\lambda^2}-4\frac{\mu}{\lambda}}}{2}$$

(3.14)

and

$$b^2 - a^2 = \frac{-\mu}{\lambda}$$

(3.15)

Thus Equation (3.12) becomes

$$\frac{1}{\left(s-(a-b)\right)\left(s-(a+b)\right)} = \frac{A}{\left(s-(a-b)\right)} + \frac{B}{\left(s-(a+b)\right)}$$

(3.16)

Which on solving we have

$$A = -\frac{1}{2b}$$

and

$$B = \frac{1}{2b}.$$

Thus on substitution and integration of Equation (3.11) we have

$$\frac{s-(a+b)}{s-(a-b)} = Ke^{-2b\lambda t}$$

(3.17)

Where K is a constant of integration.

Making s the subject we have

$$s = \frac{(a+b) - K(a-b)e^{-2b\lambda t}}{1-Ke^{-2b\lambda t}}$$

(3.18)

Making K the subject we obtain

$$K = \frac{s-(a+b)}{s-(a-b)}e^{2b\lambda t}$$

(3.19)

By solving the second part of the Equality of Equation (3.9)

$$\frac{dG(.)}{dt}=\left(\nu-\frac{\alpha}{s}\right)(s-1)G(.)$$

(3.20)

On integration we have

$$\int\frac{dG(.)}{G(.)}=\ln G(s,t)=\int\left(\nu s-\nu-\alpha+\frac{\alpha}{s}\right)dt+C$$

(3.21)

$$G(.)=Ce^{\int_0^t\left(\nu s-\nu-\alpha+\frac{\alpha}{s}\right)dt}$$

Where C is the constant of integration.

At $t=0$, $G(s,0)=C$ therefore,

$$G(s,t)=G(s,0)\exp\int_0^t\left(\nu s-\nu-\alpha+\frac{\alpha}{s}\right)dt$$

(3.22)

And from Equation (3.7) we have

$$G(s,0)=\sum_{n=0}^{\infty}\Gamma_n s^n$$

And thus Equation (3.22) becomes

$$G(s,t)=\sum_{n=0}^{\infty}\Gamma_n s^n\exp\int_0^t\left(\nu s-\nu-\alpha+\frac{\alpha}{s}\right)dt$$

(3.23)

Which upon using Equation (3.18)

$$G(s,t)=\exp\left[\begin{array}{l}\underbrace{\nu\int_0^t\left(\frac{(a+b)-K(a-b)e^{-2b\lambda\tau}-1+Ke^{-2b\lambda\tau}}{1-Ke^{-2b\lambda\tau}}\right)d\tau}_{=d}\\ \underbrace{+\alpha\int_0^t\left(\frac{1-Ke^{-2b\lambda\tau}-(a+b)+K(a-b)e^{-2b\lambda\tau}}{(a+b)-K(a-b)e^{-2b\lambda\tau}}\right)d\tau}_{=h}\end{array}\right]\times\sum_{n=0}^{\infty}\Gamma_n s^n$$

(3.24)

Solving d we have

$$d = \nu\int_0^t \left(\frac{(a+b) - K(a-b)e^{-2b\lambda\tau} - 1 + Ke^{-2b\lambda\tau}}{1 - Ke^{-2b\lambda\tau}} \right) d\tau$$

which can be rearranged into

$$d = \underbrace{\int_0^t \frac{\nu(a+b-1)}{1-Ke^{-2b\lambda\tau}} d\tau}_{=d_1} + \underbrace{\int_0^t \frac{\nu(1+b-a)Ke^{-2b\lambda\tau}}{1-Ke^{-2b\lambda\tau}} d\tau}_{=d_2}$$

(3.25)

Using the identity

$$\int \frac{dt}{c+be^{at}} = \frac{1}{ac}\log\frac{be^{at}}{c+be^{at}} + \text{constant}$$

(3.26)

In the integration of Equation (3.25), we have

$$d_1 = \frac{\nu(a+b-1)}{2b\lambda}\ln\left[\frac{1-Ke^{-2b\lambda}}{(1-K)e^{-2b\lambda}}\right],$$

(3.27)

$$d_2 = \frac{\nu(1+b-a)}{2b\lambda}\ln\left[\frac{1-Ke^{-2b\lambda}}{(1-K)}\right]$$

(3.28)

Combining d_1 and d_2 we have

$$d = d_1 + d_2 = \ln\left(\left[\frac{1-Ke^{-2b\lambda}}{1-K}\right]^{\frac{\nu}{\lambda}} e^{\nu(a+b-1)t} \right)$$

(3.29)

Solving part h of Equation (3.24) we have

$$h = \alpha\int_0^t \left(\frac{1-Ke^{-2b\lambda\tau} - (a+b) + K(a-b)e^{-2b\lambda\tau}}{(a+b) - K(a-b)e^{-2b\lambda\tau}} \right) d\tau$$

(3.30)

Which can be split into

$$h=\underbrace{\alpha\int_0^t \frac{1-(a+b)}{a+b-K(a-b)e^{-2b\lambda\tau}}d\tau}_{=h_1}+\underbrace{\alpha\int_0^t \frac{K(a-b-1)}{-K(a-b)+(a+b)e^{2b\lambda\tau}}d\tau}_{=h_2}$$

(3.31)

Solving Equation (3.31) using the identity in Equation (3.26) we have;

$$h_1=\frac{\alpha(a+b-1)}{2b(a+b)\lambda}\ln\left[\frac{\left[(a+b)-K(a-b)\right]e^{-2b\lambda t}}{a+b-K(a-b)e^{-2b\lambda t}}\right]$$

(3.32)

In solving the remained part of Equation (3.31) we have

$$h_2=\frac{\alpha(a-b-1)}{2b\lambda(b-a)}\ln\left[\frac{\left[K(b-a)+(a+b)\right]e^{2b\lambda t}}{K(b-a)+(a+b)e^{2b\lambda t}}\right]$$

(3.33)

Thus we have

$$h=h_1+h_2=\ln\left[\left(\frac{(a+b)+K(b-a)}{(a+b)+K(b-a)e^{-2b\lambda t}}\right)^{\frac{\alpha}{\mu}}e^{\frac{-\alpha(a+b-1)t}{a+b}}\right]$$

(3.34)

Combining the solution for d and h as given in Equation (3.29) and (3.34) respectively we have

$$d+h=\ln\left[\left(\frac{(a+b)+K(b-a)}{(a+b)+K(b-a)e^{-2b\lambda t}}\right)^{\frac{\alpha}{\mu}}\left[\frac{1-Ke^{-2b\lambda t}}{(1-K)}\right]^{\frac{\nu}{\lambda}}e^{\frac{-\alpha(a+b-1)t}{a+b}+\nu(a+b-1)t}\right]$$

(3.35)

Hence the generating Function $G(s,\mathrm{t})$ becomes

$$G(s,t)=\exp\left[\ln\left[\left(\frac{(a+b)+K(b-a)}{(a+b)+K(b-a)e^{-2b\lambda t}}\right)^{\frac{\alpha}{\mu}}\left[\frac{(1-K)}{1-Ke^{-2b\lambda t}}\right]^{\frac{-\nu}{\lambda}}e^{\frac{-\alpha(a+b-1)t}{a+b}+\nu(a+b-1)t}\right]\times\sum_{n=0}^{\infty}\Gamma_n s^n\right]$$

$$=\left[\left(\frac{(a+b)+K(b-a)}{(a+b)+K(b-a)e^{-2b\lambda t}}\right)^{\frac{\alpha}{\mu}}\left[\frac{(1-K)}{1-Ke^{-2b\lambda t}}\right]^{\frac{-\nu}{\lambda}}\exp\left\{(a+b-1)\left(\nu-\frac{\alpha}{a+b}\right)t\right\}\right]\times\exp\sum_{n=0}^{\infty}\Gamma_n s^n$$

(3.36)

Which can be written as

$$G(s,t)=W_1^{\frac{\alpha}{\mu}}W_2^{\frac{-\nu}{\lambda}}\exp\left\{(a+b-1)\left(\nu-\frac{\alpha}{a+b}\right)t\right\}\exp\sum_{n=0}^{\infty}\Gamma_n s^n$$

(3.37)

Where

$$W_1=\left(\frac{(a+b)+K(b-a)}{(a+b)+K(b-a)e^{-2b\lambda t}}\right)=W_1(s,t)$$

and

$$W_2=\frac{1-K}{1-Ke^{-2b\lambda t}}=W_2(s,t).$$

Upon replacing K as given in Equation (3.19) into Equation (3.36) we have

$$W_1=\frac{\left(b^2-a^2+as\right)\left(1-e^{2b\lambda t}\right)}{2bs}+\frac{\left(1+e^{2b\lambda t}\right)}{2},$$

(3.38)

But $b^2-a^2=\frac{-\mu}{\lambda}$, thus Equation (3.38) becomes

$$W_1=\frac{1}{2bs}\left[\frac{-\mu}{\lambda}+as\right]\left(1-e^{2b\lambda t}\right)+\frac{1}{2}\left(1+e^{2b\lambda t}\right)$$

(3.39)

Upon substituting K into W_2 we have

$$W_2=\frac{s-a+b-\left[s-(a+b)\right]e^{2b\lambda t}}{2b}$$

(3.40)

Thus Equation (3.36) becomes

$$G(s,t) = W_1^{\frac{\alpha}{\mu}} W_2^{\frac{-\nu}{\lambda}} \exp \sum_{n=0}^{\infty} \Gamma_n s^n \exp\left\{(a+b-1)\left(\nu - \frac{\alpha}{a+b}\right)t\right\}$$

(3.41)

Where W_1 and W_2 are as given in Equations (3.39) and (3.40) respectively. Thus we have obtained the form of generating function for the distribution of the size of the population at any time t for the BDIE process with genocide parameter.

4. Derivation of the Mean and Variance

The most important moments of a distribution are the mean and variance. It is possible to find the mean and variance in sections 4.1 and 4.2 respectively.

4.1 Mean

Following the standards procedures for finding the mean we have:

$$\frac{\partial G(s,t)}{\partial s} = E\big(n(t)\big) = \bar{n}(t)$$

(4.1)

Taking natural log of both sides of Equation (3.41) we have

$$\ln G(s,t) = \frac{\alpha}{\mu}\ln W_1 - \frac{-\nu}{\lambda}\ln W_2 + \sum_{n=0}^{\infty}\Gamma_n s^n + \left\{(a+b-1)\left(\nu - \frac{\alpha}{a+b}\right)t\right\}$$

(4.2)

Which upon differentiation with respect to s gives;

$$\frac{G'}{G} = \frac{\alpha W_1'}{\mu W_1} - \frac{\nu W_2'}{\lambda W_2} + \sum_{n=0}^{\infty} n\Gamma_n s^{n-1} = H(s,t)$$

(4.3)

Where the $prime = \frac{\partial}{\partial s}$ with $G = G(s,t)$ .Thus,

$$\frac{\partial G}{\partial s} = G\left(\frac{\alpha W_1'}{\mu W_1} - \frac{\nu W_2'}{\lambda W_2} + \sum_{n=0}^{\infty} n\Gamma_n s^{n-1}\right) = G(s,t)H(s,t)$$

(4.4)

Where

$$W_1' = \frac{\mu}{2bs^2\lambda}\left(1 - e^{2b\lambda t}\right)$$

$$W_2' = \frac{1-e^{2b\lambda t}}{2b}$$

(4.5)

Setting $s=1$ we have

$$W_1\big|_{s=1} = \frac{\left(\frac{-\mu}{\lambda}+a\right)\left(1-e^{2b\lambda t}\right)}{2b} + \frac{\left(1+e^{2b\lambda t}\right)}{2}$$

$$W_1'\Big|_{s=1} = \frac{\mu}{2b\lambda}\left(1-e^{2b\lambda t}\right)$$

(4.6)

$$W_2\big|_{s=1} = \frac{1-a+b-\left(1-\left(a+b\right)\right)e^{2b\lambda t}}{2b}$$

S $W_2'\big|_{s=1} = \frac{1-e^{2b\lambda t}}{2b}$

(4.7)

Thus,

$$\bar{n}(t) = W_1^{\frac{\alpha}{\mu}} W^{\frac{-\nu}{\lambda}} \left(\frac{\alpha W_1'}{\mu W_1} - \frac{\nu W_2'}{\lambda W_2} + n_0\right) \exp\left\{\left(a+b-1\right)\left(\nu - \frac{\alpha}{a+b}\right)t\right\}$$

(4.8)

Where W_1, W_1', W_2 and W_2' are given in Equations (4.6) and (4.7)respectively while $\sum_{n=0}^{\infty} n\Gamma_n = n_0$ as shown in Equations (3.7a) and (3.7b). Hence Equation (4.8) is the stochastic population mean for the BDIE process with genocide parameter.

4.2 Variance

We can now proceed to find the variance, $\sigma^2(t)$ by differentiating Equation (3.41) twice. Thus,

$$\frac{\partial G}{\partial s} = G(s,t)H(s,t)$$

(4.9)

Taking natural log of both Equation (4.9) we get

$$\ln G' = \ln G(s,t) + \ln H(s,t)$$

(4.10)

Which upon differentiating with respect to s and denoting $H(s,t)$ by H, we obtain:

$$\frac{G''}{G'} = \frac{G'}{G} + \frac{H'}{H}$$

$$G'' = G\left[H^2 + H'\right]$$

(4.11)

Thus Equation (4.11) becomes

$$\frac{\partial^2 G(s,t)}{\partial s^2} = G(s,t)\left[H^2 + H'\right]$$

(4.12)

Where

$$H' = \frac{\alpha}{\mu}\left(\frac{W_1''W_1 - (W_1')^2}{W_1^2}\right) - \frac{\nu}{\lambda}\left(\frac{W_2''W_2 - (W_2')^2}{W_2^2}\right) + \sum_{n=0}^{\infty} n(n-1)\Gamma_n s^{n-2}.$$

Following the standard procedure for finding the variance we have;

$$\sigma^2(t) = \frac{\partial^2 G(1,t)}{\partial s^2} + \bar{n}(t) - \bar{n}^2(t)$$

$$= E\big(n(n-1)\big) + \bar{n}(t) - \bar{n}^2(t)$$

(4.13)

$$= E\left(n^2\right) - \left[E\left(\bar{n}(t)\right)\right]^2$$

(4.14)

Therefore, the stochastic population variance model for BDIE process with genocide parameter is

$$\sigma^2(t) = G\big|_{s=1}\left(H^2\big|_{s=1} + H'\big|_{s=1}\right) + \bar{n}(t) - \left[\bar{n}(t)\right]^2$$

(4.15)

Where

$$H\big|_{s=1} = \frac{\alpha W_1'}{\mu W_1} + \frac{\nu W_2'}{\lambda W_2} + n_0$$

$$H'\big|_{s=1} = \frac{\alpha}{\mu}\left(\frac{W_1'' W_1' - \left(W_1'\right)^2}{W_1^2}\right) - \frac{\nu}{\lambda}\left(\frac{W_2'}{W_2}\right)^2 + \sigma_0{}^2 + n_0 - n_0{}^2$$

$$\bar{n}\left(t\right) = W_1^{\frac{\alpha}{\mu}} W^{\frac{-\nu}{\lambda}}\left(\frac{\alpha W_1'}{\mu W_1} - \frac{\nu W_2'}{\lambda W_2} + n_0\right)\exp\left\{\left(a+b-1\right)\left(\nu - \frac{\alpha}{a+b}\right)t\right\}.$$

Additionally, W_1' and W_2' is as shown in Equation (4.6) and (4.7) respectively while

$$W_1'' = \frac{-\mu}{b\lambda}\left(1 - e^{2b\lambda t}\right),$$

$$W_2'' = 0,$$

and

$$\sum_{n=0}^{\infty} n\left(n-1\right)\Gamma_n = \sigma_0{}^2 + n_0 - n_0{}^2$$, using Equation **(4.13)**.

Thus, Equation (4.15) is the stochastic population Variance for the BDIE process with genocide incidence. Next we will apply the developed stochastic models into the real population data.

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