Bayesian and E-Bayesian Method of Estimation of Parameter of Rayleigh Distribution- A Bayesian Approach under Linex Loss Function

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Abstract

In this paper, Bayesian and E-Bayesian method of estimation are proposed for estimating the parameter of Rayleigh distribution. The Bayes estimate of the parameter is derived under the assumption that the prior distribution is informative i.e. gamma prior using Linex loss function. Further, comparison between the E-Bayes estimators with the associated Bayes estimators have been carried out through simulation study using MATLAB software.

Keywords: Rayleigh distribution, Linex Loss function, Bayes and E-Bayes estimators, Gamma prior.

1. INTRODUCTION

The Rayleigh distribution is a continuous probability distribution serving as a special case of the well-known Weibull distribution. This distribution has long been considered to have significant applications in fields such as survival analysis, reliability theory and especially communication engineering. The Rayleigh distribution provides a population model which is useful in several areas of statistics (Fernandez, 2000). In the literature, many researcher studied properties of the Rayleigh distribution, particularly in life testing and reliability.

When considering the complete Rayleigh model, the probability density function is given by

\[ f(x, \theta) = 2 \theta x e^{-\theta x^2}, x, \theta > 0; \]  

(1)

using the parameterization of the distribution as proposed by Bhattacharya and Tyagi.
(1990), and is denoted by \( X \sim \text{Rayleigh}(\theta) \). The parameter \( \theta \) is a scale parameter, and characterizes the lifetime of the object under consideration in application.

Many authors have studied Rayleigh distribution e.g. Ferreira et al. (2016) have proposed Bayes estimators including shrinkage estimators of the unknown parameter of the censored Rayleigh distribution using Al-Bayyati loss function considering different objective prior distribution. Dey (2012) has obtained Bayes estimator of parameter and reliability function of inverse Rayleigh distribution under two loss functions and also obtained associated risk functions of the Bayes estimator. Sanat et al. (2016) have proposed Bayes estimate of parameter of Rayleigh distribution using Quasi prior under different loss functions. Ahmed et al. (2013) have obtained Bayes estimate of parameter of Rayleigh distribution using Jeffrey’s and extension of Jeffrey’s prior under Squared error and Al-Bayyati’s loss function.

The main objective of this paper is to introduce a statistical comparison between the Bayesian and Expected-Bayesian procedures for estimating the parameter of Rayleigh distribution. The resulting estimators are obtained by using Linex loss function.

The layout of the paper is as follow. In Section 2, Bayes estimate of parameter have been obtained using conjugate prior under Linex loss function. In Section 3, E-Bayes estimate have also been obtained using three different prior distributions. Finally, comparison between Bayes and E-Bayes estimates have been made using simulation study in Section 4. Some concluding remarks have been given in Section 5.

2. BAYESIAN ESTIMATION.

In this section, Bayes estimate of parameter of Rayleigh distribution is obtained by using Linex loss function. Let \( X_1, X_2, X_3, \ldots \) be a sequence of random variables from Rayleigh distribution, whose density function is given by (1), then the likelihood function is given by

\[
L(\mathbf{x}, \theta) = (2\theta)^n \prod_{i=1}^{n} x_i \cdot e^{-\theta \sum_{i=1}^{n} x_i^2}
\]

We use the gamma conjugate prior density for the parameter \( \theta \) and the pdf of gamma prior density with scale parameter \( r \) is given by

\[
g(\theta / r) = \frac{e^{-\theta} \cdot \theta^{r-1}}{\Gamma(r)}; \quad r > 0, \theta > 0
\]

On combining (2) and (3), and using Bayes theorem, the posterior density of \( \theta \) given \( \mathbf{x} \) is given by

\[
P\left(\theta / \mathbf{x}\right) \propto L(\mathbf{x}, \theta) \cdot g(\theta / r)
\]

\[
= (2\theta)^n \prod_{i=1}^{n} x_i \cdot e^{-\theta \sum_{i=1}^{n} x_i^2} \cdot \frac{e^{-\theta} \cdot \theta^{r-1}}{\Gamma(r)}; \quad r > 0, \theta > 0, \mathbf{x} > 0
\]
Bayesian and E-Bayesian Method of Estimation of Parameter of Rayleigh

\[ P(\theta|x) = \frac{e^{-\theta \sum_{i=1}^{n}(x_i)^2 + 1} \cdot \theta^{n+r-1} \cdot \prod_{i=1}^{n}(x_i)^2 + 1}{\Gamma(n+r)}; x > 0, \theta > 0, r > 0 \]  

(4)

Bayesian estimation of \( \theta \) under Linex loss function.

Zellner represent Linex (i.e. linear exponential) loss function as
\[ L(\theta, \hat{\theta}) = a \{ \exp[b(\hat{\theta} - \theta) - b(\hat{\theta} - \theta) - 1] \}; \]
where \( a > 0, b \neq 0 \); a is scale of loss function and b determines its shape. Without loss of generality, we assume \( a = 1 \) and obtain Bayes estimate of \( \theta \).

Thus Bayes estimator of \( \theta \) under Linex loss function is given by

\[ \hat{\theta}_{BL} = \frac{n+r}{b} \cdot \log \left[ \frac{b + \sum_{i=1}^{n}(x_i)^2 + 1}{\sum_{i=1}^{n}(x_i)^2 + 1} \right] \]  

(5)

3. EXPECTED - BAYESIAN ESTIMATE OF PARAMETER USING LINEX LOSS FUNCTION

Han (2007) introduced a new method, named E-Bayesian method to estimate failure probability. The E-Bayes estimate of \( \theta \) i.e. expectation of the Bayes estimate of \( \theta \) is given by
\[ \hat{\theta}_{EB} = \mathbb{E}(\theta|x) = \int_{Q} \hat{\theta}_{BE} \pi(\theta, r) * dr \]
where \( Q \) is the domain of \( r \) for which the prior density is decreasing in \( \theta \). \( \hat{\theta}_{BE} \) is the Bayes estimate of \( \theta \) under the Linex loss function. In order to obtain E-Bayes estimates of \( \theta \), we have to choose prior distribution of hyper parameter \( r \). These distributions are used to study the impact of different prior distributions on E-Bayes estimation of \( \theta \). The following distributions of \( r \) are given by
\[ \pi_1(\theta, r) = \frac{2(c-r)}{c^2}; \quad 0 < r < c, \]  

(6)
$$\pi_2(\theta, r) = \frac{1}{c}; \quad 0 < r < c,$$

$$\pi_3(\theta, r) = \frac{2r}{c^2}; \quad 0 < r < c,$$

(7) (8)

**E-Bayesian Estimation of \( \theta \) under Linex loss function:**

E-Bayesian estimate of \( \theta \) relative to Linex loss function based on \( \pi_1(\theta, r) \), is denoted by \( \hat{\theta}_{EBL_1} \) and is obtained by using (5) and (6).

$$\hat{\theta}_{EBL_1} = \int_0^c \hat{\theta}_{BE} \ast \pi_1(\theta, r) \ast dr$$

$$= \int_0^c \left( \frac{n + r}{b} \ast \log \left( \frac{b + \sum_{i=1}^{n}(x_i)^2 + 1}{\sum_{i=1}^{n}(x_i)^2 + 1} \right) \right) \ast \frac{2(c - r)}{c^2} \ast dr$$

$$= \left[ \frac{3n + c}{3b} \right] \ast \log \left( \frac{b + \sum_{i=1}^{n}(x_i)^2 + 1}{\sum_{i=1}^{n}(x_i)^2 + 1} \right)$$

(9)

E-Bayesian estimates of \( \theta \) based on \( \pi_2(\theta, r) \), is denoted by \( \hat{\theta}_{EBL_2} \) and is obtained by using (5) and (7).

$$\hat{\theta}_{EBL_2} = \left[ \frac{2n + c}{2b} \right] \ast \log \left( \frac{b + \sum_{i=1}^{n}(x_i)^2 + 1}{\sum_{i=1}^{n}(x_i)^2 + 1} \right)$$

(10)

E-Bayesian estimates of \( \theta \) based on \( \pi_3(\theta, r) \), is denoted by \( \hat{\theta}_{EBL_3} \) and is obtained by using (5) and (8).

$$\hat{\theta}_{EBL_3} = \left[ \frac{3n + 2c}{3b} \right] \ast \log \left( \frac{b + \sum_{i=1}^{n}(x_i)^2 + 1}{\sum_{i=1}^{n}(x_i)^2 + 1} \right)$$

(11)

## 4. SIMULATION STUDY

In order to compare the performance of Bayes and E-Bayes techniques of estimation, a simulation study was conducted using Matlab software for different sample sizes and for different values of loss parameter. The following steps were conducted:

a. For given value of the prior parameter \( c \), we generate samples to find value of \( r \) from uniform priors (6-8) respectively.

b. For given value of \( r \), we generate \( \theta \) from the gamma prior density (3),

c. For known values of \( \theta \), we generate sample from Rayleigh distribution with pdf (1), and the Bayes and Expected Bayes estimates using Linex loss function are computed from (5), (9), (10) and (11) respectively.

d. The above steps are repeated 500 times and the mean square error of the Bayes and E-Bayes estimates are computed and the results are shown in table 1.
Table 1: Averaged values of MSE for Bayes and E-Bayes estimates of the parameter $\theta$.

<table>
<thead>
<tr>
<th>n (sample size)</th>
<th>$\hat{\theta}_{BL}$</th>
<th>$\hat{\theta}_{EBL_1}$</th>
<th>$\hat{\theta}_{EBL_2}$</th>
<th>$\hat{\theta}_{EBL_3}$</th>
<th>$\hat{\theta}_{EBL_1}$</th>
<th>$\hat{\theta}_{EBL_2}$</th>
<th>$\hat{\theta}_{EBL_3}$</th>
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<td>0.1541</td>
<td>0.1541</td>
<td>0.1539</td>
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<td>0.1580</td>
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<td>0.1579</td>
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<td>0.1577</td>
<td>0.1576</td>
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<td>0.1248</td>
<td>0.1115</td>
<td>0.1114</td>
<td>0.1112</td>
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</table>

5. CONCLUSION

In this paper, Bayes and Expected-Bayes methods are used for estimation of parameter of Rayleigh distribution using Linex loss function. It has been noticed from the results of simulation study, that the E-Bayes estimates have smaller Mean Square Error as compared with the associated Bayes estimate. It has also been observed that E-Bayes estimate, in most cases, tend to be more efficient than Bayes estimate except (for $n=35$, $b=0.75$ and $n=30, 40, b=-0.75$) where Bayes and E-Bayes are equally efficient.

REFERENCES


