Transient analysis of bulk arrival general service retrial queueing system with priority, Bernoulli feedback, collisions, orbital search, modified Bernoulli vacation, random breakdown and delayed repair

G. Ayyappan¹, P. Thamizhselvi

Department of Mathematics,
Pondicherry Engineering College, Puducherry, India.

Abstract

This paper deals with the analysis of two classes of batch arrivals, high-priority and non-priority(retrial) customers with non-preemptive priority service, Bernoulli Feedback, Collisions of low priority customers, Orbital search, Modified Bernoulli Vacation for an unreliable server, which consists of a break-down and delay period to start repair. Here we assume that customers arrive according to compound Poisson process. The arriving high-priority customers finds the server is busy then they form queue and then served in accordance with FCFS discipline. The arriving low-priority customers on finding the server busy then they are enter in to the orbit in accordance with FCFS retrial policy. After completion of the service if the customer(high priority/low priority) is dissatisfied with his service they can join the tail of the queue/orbit as a feedback customer or they leave the system forever if the service is satisfied. We further assume that the server may take a vacation of random length just after serving the last customer in the high-priority unit present in the system. After completion of vacation if there are no customers in the high-priority queue the server searches for the customers in the orbit or remains idle. While the server serving the customer it may breakdown at any instant and the server will be down for a short interval of time, once the server fails it send to repair process but the repair time do not start immediately, there is a delay time to start repair. The retrial time, service time, vacation time, delay time and repair time are all follows general (arbitrary) distribution and the breakdown time follows exponential distribution.

¹Corresponding author.
The time dependent probability generating functions have been obtained in terms of their Laplace transforms.

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1. **Introduction**

The study on queuing models have become an indispensable area due to its wide applicability in real life situations, all the models are considered that the units are proceed to service on a First-come First-served basis. This is obviously not only the manner of service, and there are many alternatives, such as Last-come First-served, selection in random order and selection by priority. In order to offer different quality of service for different kinds of customers, we often control a queuing system by priority mechanism. This phenomenon is common in practice. For example, in telecommunication transfer protocol, for guaranteeing different layers service for different customers, priority classes control may appear in header of IP package or in ATM cell. Priority control is also widely used in production practice, transportation management.

Retrial queues are characterized by the feature that arriving customers who find the server is busy they join the retrial group to try their luck again after a time period. Queues in which customers are allowed to conduct retrials have been extensively used to model many problems in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processor unit.

Several author had studied a single server retrial queue with orbital search and some of them had studied retrial with collisions. Ayyappan. G et al. (2009) have studied an $M/M/1$ retrial queueing system with non preemptive priority service and single vacation exhaustive service, Atencia. I. et al. (2005) have studied a single-server retrial queue with general retrial times and non-preemptive priority service, Gautam Choudhury. et al. (2012) have studied a batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair, Gomez Corral. A (1999) have studied a stochastic analysis of a single server retrial queue with general retrial times, Jain. M. et al. (2008) have studied a bulk arrival retrial queue with unreliable server and priority subscribers, Kirupa. K. et al. (2010) have studied a single-server retrial queuing system with two different vacation policies, Krishnakumar. B. et al. (2010) have studied a single server feedback retrial queue with collisions, Madan. K.C. (2011). have studied a non-preemptive priority queueing system with a single server serving two queues $M/G/1$ and $M/D/1$ with optional server vacations based on exhaustive service of the priority units, Peishu Chen. et al. (2016) have studied a batch arrival retrial $G$-queue with orbital search and non-persistent customers and Yang et al. (1994) have studied an approximation method for the $M/G/1$ retrial queue with general retrial times.

In this paper we deals with the analysis of two classes of batch arrivals that is high-priority and non-priority (retrial) customers with non-preemptive priority service,
Bernoulli feedback, collisions of low priority customers, orbital search, modified Bernoulli vacation for an unreliable server, the server may break-down at any instant and there is a delay period to start repair. Here we assume that both type of customers arrive according to compound Poisson process. The priority customers that finds the server busy are queued and then served in accordance with FCFS discipline. The arriving low-priority customers on finding the server busy then they are enter in to the orbit in accordance with FCFS retrial policy. After completion of the service if the customer (high priority/low priority) is dissatisfied with his service they can join the tail of the queue/orbit as a feedback customer or they leave the system forever if the service is satisfied. We further assume that the server may take a vacation of random length just after serving the last customer in the high-priority unit present in the system. After completion of vacation if there are no customer in the high-priority queue the server searches for the customer in the orbit or remains idle. While the server serving the customer it may breakdown at any instant, the server will be down for a short interval of time, once the server fails it send to repair process, the repair time do not start immediately, there is a delay time to start repair. The retrial time, service time, vacation time, delay time and repair time are all follows general (arbitrary) distribution and breakdown time follows exponential distribution. The time dependent probability generating functions have been obtained in terms of their Laplace transforms.

The rest of the paper is organized as follows: Mathematical description of our model in section (2). Definitions, equations governing of our model and the time dependent solution have been obtained in section (3) and (4).

2. Mathematical description of our model

(i) High Priority and Low-priority units arrive at the system in batches of variable sizes in a compound Poisson process and they are provided one by one on a FCFS basis. Let $\lambda_1 c_i dt$ and $\lambda_2 c_i dt$ ($i = 1, 2, 3, \ldots$) be the first order probability that a batch of i customers arrive at the system during a short interval of time $(t, t+dt)$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\lambda_1 > 0$, $\lambda_2 > 0$ are the average arrival rates of high-priority and low-priority customers and high-priority customers only form queue, if the server is busy. The server must serve all the high-priority units present in the system before taking up low-priority unit for service. In other words, there is no high-priority unit present in the system at the time of starting the service of a low-priority unit. Further, we assume that the server follows a non-preemptive priority rule, which means that if one or more high-priority units arrive during the service time of a low-priority unit, the current service of a low-priority unit is not stopped and a high-priority unit will be taken up for service only after the current service of a low-priority unit is completed.

(ii) Low-priority customers are considered as retrial customers. It is assumed that there is no waiting space for low priority customer and on arrival, a customer proceeds
to the server with probability $p_2$ or enters into the orbit with probability $q_2$. If the server is busy with the low-priority customer, the arriving low-priority customer collides with the customer in service resulting in both being shifted to the orbit with probability $p_2$. The retrial time, that is time between successive repeated attempts of each customer in the orbit is assumed to be generally distributed with distribution function $A(x)$, density function $a(x)$. The conditional completion rate for retrials is given by $\eta(x) = \frac{a(x)}{(1 - A(x))}$.

(iii) Each customer under high-priority and low-priority service provided by a single server. The service time for both high-priority and low-priority units follows general (arbitrary) distribution with distribution functions $B_i(v)$ and the density functions $b_i(v)$, $i = 1, 2$.

(iv) Let $\mu_i(x) dx$ be the conditional probability of completion of the high-priority and low-priority unit service during the interval $(x, x + dx]$, given that the elapsed service time is $x$, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \ i = 1, 2.$$  

and therefore,

$$b_i(v) = \mu_i(v)e^{-\int_0^v \mu_i(x) dx}, \ i = 1, 2.$$  

(v) We further assume that as soon as the service of the last high-priority unit present in the system is completed, the server has the option to take a vacation of random length with probability $\theta$, in which case the vacation starts immediately or else with probability $(1 - \theta)$, he may decide to continue serving the low-priority units present in the system, if any. In the later case, if there are no low-priority units present in the system, the server remains idle in the system waiting for the new units to arrive.

(vi) The vacation time follows general (arbitrary) distribution with distribution function $V(s)$ and the density function $v(s)$. Let $\gamma(x) dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is $x$, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)},$$  

and, therefore,

$$v(s) = \gamma(s)e^{-\int_0^s \gamma(x) dx}.$$  

(vii) The server may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Whenever the
server breaks down, its repairs do not start immediately and there is a delay time to start repair.

(viii) The delay time to start repair(failure during the service time of high priority/low priority) follows general (arbitrary) distribution with distribution function $D_i(t)$ and the density function $d_i(t)$, $i = 1, 2$. Let $\xi_i(x) dx$ be the conditional probability of completion of delay time to start repair during the interval $(y, y + dy]$, given that the elapsed delay time to start repair is $y$, so that

$$\xi_i(y) = \frac{d_i(y)}{1 - D_i(y)}, i = 1, 2$$

and therefore,

$$d_i(t) = \xi_i(t) e^{-\int_{0}^{s} \xi_i(y) dy}, i = 1, 2.$$  

(ix) The repair time follows general (arbitrary) distribution with distribution function $R_i(t)$ and the density function $r_i(t)$, $i = 1, 2$. Let $\beta_i(y) dy$ be the conditional probability of completion of repair during the interval $(y, y + dy]$, given that the elapsed repair time is $y$, so that

$$\beta_i(y) = \frac{r_i(y)}{1 - R_i(y)}, i = 1, 2$$

and therefore,

$$r_i(t) = \beta_i(t) e^{-\int_{0}^{s} \beta_i(y) dy}, i = 1, 2.$$  

(x) As soon as the completion of a service for a high-priority and low-priority customers, if they are not satisfied with their service they join the tail of the queue(orbit) as a feedback customer with probability $p_1$ and $p_3$ or may leave the system with probability $q_1$ and $q_3$ for satisfied customers.

(xi) At the completion of a vacation, the server searches for the customers in the orbit(if any), if the priority queue is empty, with probability $r$ or remains idle with complementary probability $(1 - r)$. The search time is negligible.

(xii) Various stochastic processes involved in the system are assumed to be independent of each other.

3. Definitions and notations

We define
(i) $P_{m,n}(x, t) =$ Probability that at time $t$, the server is active providing service and there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit excluding the one high-priority unit in service with elapsed service time for this customer is $x$. Accordingly, $P_{m,n}(t) = \int_0^\infty P_{m,n}(x, t) \, dx$ denotes the probability that at time $t$ there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit excluding one high-priority unit in service without regard to the elapsed service time $x$ of a high-priority unit.

(ii) $q_{m,n}(x, t) =$ Probability that at time $t$, the server is active providing service and there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit excluding the one low-priority unit in service with elapsed service time for this customer is $x$. Accordingly, $q_{m,n}(t) = \int_0^\infty q_{m,n}(x, t) \, dx$ denotes the probability that at time $t$ there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit excluding one low-priority unit in service without regard to the elapsed service time $x$ of a low-priority unit.

(iii) $V_{m,n}(x, t) =$ Probability that at time $t$, the server is on vacation with elapsed vacation time $x$ and there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit. Accordingly, $V_{m,n}(t) = \int_0^\infty V_{m,n}(x, t) \, dx$ denotes the probability that at time $t$ there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed vacation time is $x$.

(iv) $D^{(i)}_{m,n}(x, y, t) =$ Probability that at time $t$, the server is on delay to start repair(server inactive during the service time of high priority/low priority) with elapsed delay time is $y$ and there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit. Accordingly, $D^{(i)}_{m,n}(x, t) = \int_0^\infty D^{(i)}_{m,n}(x, y, t) \, dy$ denotes the probability that at time $t$ there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed delay time is $y$, $i = 1, 2$.

(v) $R^{(i)}_{m,n}(x, y, t) =$ Probability that at time $t$, the server is on repair with elapsed repair time is $y$ and there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit. Accordingly, $R^{(i)}_{m,n}(x, t) = \int_0^\infty R^{(i)}_{m,n}(x, y, t) \, dy$ denotes the probability that at time $t$ there are $m(m \geq 0)$ high-priority units in the queue and $n(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed repair time is $y$, $i = 1, 2$.

(vi) $I_{0,0}(t) =$ Probability that at time $t$, there are no high-priority and low-priority customers in the system and the server is idle but available in the system.
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4. Equations Governing the System

\[
\frac{d}{dt} I_{(0,0)}(t) = -(\lambda_1 + \lambda_2)I_{(0,0)}(t) + q_1(1 - \theta) \int_0^{\infty} P_{0,0}(x, t) \mu_1(x) dx \\
+ q_3 \int_0^{\infty} q_{0,0}(x, t) \mu_2(x) dx + \int_0^{\infty} V_{0,0}(x, t) \gamma(x) dx,
\]

(1)

\[
\frac{\partial}{\partial t} I_{(0,n)}(x, t) + \frac{\partial}{\partial x} I_{(0,n)}(x, t) = -(\lambda_1 + \lambda_2 + \eta(x))I_{(0,n)}(x, t); \quad n \geq 1,
\]

(2)

\[
\frac{\partial}{\partial t} P_{m,n}(x, t) + \frac{\partial}{\partial x} P_{m,n}(x, t) = -(\lambda_1 + \lambda_2 + \mu_1(x) + \alpha)P_{m,n}(x, t) \\
+ \int_0^{\infty} R_{1,m,n}(x, y, t) \beta_1(y) dy \\
+ \lambda_1 \sum_{i=1}^{m} C_i P_{m-i,n}(x, t) \\
+ \lambda_2 \sum_{i=1}^{n} C_i P_{m,n-i}(x, t); \quad m, n \geq 0,
\]

(3)

\[
\frac{\partial}{\partial t} q_{m,n}(x, t) + \frac{\partial}{\partial x} q_{m,n}(x, t) = -(\lambda_1 + \lambda_2 + \mu_2(x) + \alpha)q_{m,n}(x, t) \\
+ \int_0^{\infty} R_{2,m,n}(x, y, t) \beta_2(y) dy \\
+ \lambda_1 \sum_{i=1}^{m} C_i q_{m-i,n}(x, t) \\
+ \lambda_2 (1 - p_2) \sum_{i=1}^{n} C_i q_{m,n-i}(x, t); \quad m, n \geq 0,
\]

(4)

\[
\frac{\partial}{\partial t} V_{m,n}(x, t) + \frac{\partial}{\partial x} V_{m,n}(x, t) = -(\lambda_1 + \lambda_2 + \gamma(x))V_{m,n}(x, t) \\
+ \lambda_1 \sum_{i=1}^{m} C_i V_{m-i,n}(x, t) \\
+ \lambda_2 \sum_{i=1}^{n} C_i V_{m,n-i}(x, t); \quad m, n \geq 0.
\]

(5)
\[
\frac{\partial}{\partial t} D_{i,m,n}(x,y,t) + \frac{\partial}{\partial x} D_{i,m,n}(x,y,t) = -(\lambda_1 + \lambda_2 + \xi_i(y)) D_{i,m,n}(x,y,t) \\
+ \lambda_1 \sum_{i=1}^{m} C_i D_{i,m-i,n}(x,y,t) \\
+ \lambda_2 \sum_{i=1}^{n} C_i D_{i,m,n-i}(x,y,t);
\]
\[i = 1, 2, m, n \geq 0,
\]
\(6\)

\[
\frac{\partial}{\partial t} R_{i,m,n}(x,y,t) + \frac{\partial}{\partial x} R_{i,m,n}(x,y,t) = -(\lambda_1 + \lambda_2 + \beta_i(y)) R_{i,m,n}(x,y,t) \\
+ \lambda_1 \sum_{i=1}^{m} C_i R_{i,m-i,n}(x,y,t) \\
+ \lambda_2 \sum_{i=1}^{n} C_i R_{i,m,n-i}(x,y,t);
\]
\[i = 1, 2, m, n \geq 0.
\]
\(7\)

The above set of equations are to be solved under the following boundary conditions at \(x = 0, y = 0\).

\[
I(0,1)(0,t) = q_1(1-\theta) \int_{0}^{\infty} P_{0,1}(x,t) \mu_1(x)dx \\
+ q_3 \int_{0}^{\infty} q_0,1(x,t) \mu_2(x)dx \\
+ p_3 \int_{0}^{\infty} q_{0,0}(x,t) \mu_2(x)dx + (1-r) \int_{0}^{\infty} V_{0,1}(x,t) \gamma(x)dx,
\]
\(8\)

\[
I(0,n)(0,t) = q_1(1-\theta) \int_{0}^{\infty} P_{0,n}(x,t) \mu_1(x)dx \\
+ q_3 \int_{0}^{\infty} q_{0,n}(x,t) \mu_2(x)dx + p_3 \int_{0}^{\infty} q_{0,n-1}(x,t) \mu_2(x)dx \\
+ (1-r) \int_{0}^{\infty} V_{0,n}(x,t) \gamma(x)dx + \lambda_2 p_2 \int_{0}^{\infty} q_{0,n-2}(x,t)dx,
\]
\(9\)

\[
P_{m,0}(0,t) = \lambda_1 C_{m+1} I(0,0)(t) + q_1 \int_{0}^{\infty} P_{m+1,0}(x,t) \mu_1(x)dx \\
+ p_1 \int_{0}^{\infty} P_{m,0}(x,t) \mu_1(x)dx \\
+ q_3 \int_{0}^{\infty} q_{m+1,0}(x,t) \mu_2(x)dx + \int_{0}^{\infty} V_{m+1,0}(x,t) \gamma(x)dx; m \geq 0,
\]
\(10\)
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\[ P_{m,0}(0, t) = \lambda_1 C_{m+1} I_{(0,1)}(t) + q_1 \int_0^\infty P_{m+1,1}(x, t) \mu_1(x) \, dx \]
\[ + p_1 \int_0^\infty P_{m,1}(x, t) \mu_1(x) \, dx \]
\[ + q_3 \int_0^\infty q_{m+1,1}(x, t) \mu_2(x) \, dx + p_3 \int_0^\infty q_{m+1,0}(x, t) \mu_2(x) \, dx \]
\[ + \int_0^\infty V_{m+1,1}(x, t) \gamma(x) \, dx; \quad m \geq 0, \quad (11) \]

\[ P_{m,n}(0, t) = \lambda_1 C_{m+1} I_{(0,n)}(t) + q_1 \int_0^\infty P_{m+1,n}(x, t) \mu_1(x) \, dx \]
\[ + p_1 \int_0^\infty P_{m,n}(x, t) \mu_1(x) \, dx \]
\[ + q_3 \int_0^\infty q_{m+1,n}(x, t) \mu_2(x) \, dx + p_3 \int_0^\infty q_{m+1,n-1}(x, t) \mu_2(x) \, dx \]
\[ + \int_0^\infty V_{m+1,n}(x, t) \gamma(x) \, dx \]
\[ + \lambda_2 p_2 \sum_{i=1}^{n-1} C_{n-i} \int_0^\infty q_{m+1,i-1}(x, t) \, dx; \quad m \geq 0, n \geq 1, \quad (12) \]

\[ q_{0,0}(0, t) = \lambda_2 p_2 C_1 I_{(0,0)}(t) + \int_0^\infty I_{0,1}(x, t) \eta(x) \, dx \]
\[ + r \int_0^\infty V_{0,1}(x, t) \gamma(x) \, dx, \quad (13) \]

\[ q_{0,n}(0, t) = \lambda_2 p_2 C_{n+1} I_{(0,0)}(t) + \int_0^\infty I_{0,n}(x, t) \eta(x) \, dx \]
\[ + r \int_0^\infty V_{0,n+1}(x, t) \gamma(x) \, dx \]
\[ + \lambda_2 p_2 \sum_{i=1}^n C_i \int_0^\infty I_{0,n-i}(x, t) \, dx; \quad n \geq 1, \quad (14) \]

\[ V_{0,n}(0, t) = q_1 \theta \int_0^\infty P_{0,n}(x, t) \mu_1(x) \, dx; \quad n \geq 0, \quad (15) \]

\[ D_{1,m,n}(0, t) = \alpha \int_0^\infty P_{m,n}(x, t) \, dx; \quad m, n \geq 0, \quad (16) \]
\[ D_{2,m,n}(x, 0, t) = \alpha \int_0^\infty q_{m,n}(x, t)\,dx; m, n \geq 0, \quad (17) \]

\[ R_{1,m,n}(x, 0, t) = \int_0^\infty D_{1,m,n}(x, y, t)\xi_1(y)\,dy; m, n \geq 0. \quad (18) \]

\[ R_{2,m,n}(x, 0, t) = \int_0^\infty D_{2,m,n}(x, y, t)\xi_2(y)\,dy; m, n \geq 0. \quad (19) \]

We assume that initially there are no customers in the system so that the server is idle.

\[ I_{0,0}(0) = 1, P_{m,n}(0) = q_{m,n}(0) = D_{i,m,n}(0) = R_{i,m,n}(0) = \eta(t) = 0; m, n \geq 0, i = 1, 2. \quad (20) \]

Next, we define the following probability generating functions:

\[ I_0(0, z_2, t) = \sum_{n=1}^\infty I_{0,n}(0, t), \]

\[ A(x, z_1, z_2, t) = \sum_{m,n=0}^\infty \sum_{n=0}^\infty z_1^n z_2^n A_{m,n}(x, t) \quad (21) \]

where \( A = P, q, V, D_i, R_i; i = 1, 2 \), which are convergent inside the circle given by \(|z_1| \leq 1, |z_2| \leq 1\). Taking Laplace transform from equations (1) to (19) and then solve it we get

\[ \bar{T}(0)(x, z_2, s) = \bar{T}(0)(0, z_2, s)e^{-\left(s+\lambda_1+\lambda_2\right)x-\int_0^x \eta(t)\,dt}, \quad (22) \]

\[ \bar{P}(x, z_1, z_2, s) = \bar{P}(0, z_1, z_2, s)e^{-\phi_1(z, s)x-\int_0^x \mu_1(t)\,dt}, \quad (23) \]

\[ \bar{q}(x, z_1, z_2, s) = \bar{q}(0, z_1, z_2, s)e^{-\phi_2(z, s)x-\int_0^x \mu_2(t)\,dt}, \quad (24) \]

\[ \bar{V}(x, z_1, z_2, s) = \bar{V}(0, z_1, z_2, s)e^{-A(z, s)x-\int_0^x \gamma(t)\,dt}, \quad (25) \]

\[ \bar{D}_i(x, y, z_1, z_2, s) = \bar{D}_i(x, 0, z_1, z_2, s)e^{-\left(s+\lambda_i[1-C(z_1)]+\lambda_2[1-C(z_2)]\right)y-\int_0^y \xi_i(y)\,dy} \quad (26) \]
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\[ \overline{R}_t(x, y, z_1, z_2, s) = \overline{R}_t(x, 0, z_1, z_2, s) e^{-\left(\gamma + \lambda_1[1-C(z_1)] + \lambda_2[1-C(z_2)]\right)y} - \int_0^y \beta_t(y) dy \]  \hspace{1cm} (27)

\[ A(z, s) = s + \lambda_1[1 - C(z_1)] + \lambda_2[1 - C(z_2)], \phi_1(z, s) = A(z, s) + \alpha[1 - D_1(A(z, s)) \overline{R}_1(A(z, s))] \]

\[ \phi_2(z, s) = A(z, s) + \alpha[1 - D_2(A(z, s)) \overline{R}_2(A(z, s))] \]

\[ \overline{q}_0(0, z_2, s) = \frac{h_2}{g^2} \]  \hspace{1cm} (28)

where

\[ h_2 = \lambda_2 p_2 C(z_2) I_{0,0} \left\{ 1 - \theta V(A_2(z, s)) + \theta V(A_1(z, s)) \right\} - \left\{ (s + \lambda_1 + \lambda_2) I_{0,0} - 1 \right\} \left\{ r \theta V(A_1(z, s)) \lambda_1 C[g(z_2)] + \left( 1 - \theta V(A_2(z, s)) + \theta V(A_1(z, s)) \right) \times \overline{M}(s + \lambda_1 + \lambda_2) + \lambda_2 p_2 C(z_2) \left[ \frac{1 - \overline{M}(s + \lambda_1 + \lambda_2)}{(s + \lambda_1 + \lambda_2)} \right] \right\} \]

\[ g_2 = z_2 \left\{ 1 - \theta V(A_2(z, s)) + \theta V(A_1(z, s)) \right\} - \lambda_1 C[g(z_2)] \left\{ 1 - \theta + (1 - r) \theta V(A_1(z, s)) \right\} - \left[ q_3 + z_2 p_3 + \lambda_2 p_2 C(z_2) \right] \left\{ r \theta V(A_1(z, s)) + \left( 1 - \theta + (1 - r) \theta V(A_1(z, s)) \right) \times \overline{M}(s + \lambda_1 + \lambda_2) + \lambda_2 p_2 C(z_2) \left[ \frac{1 - \overline{M}(s + \lambda_1 + \lambda_2)}{(s + \lambda_1 + \lambda_2)} \right] \right\} \times \left\{ B_2(\psi_1(z, s)) - B_2(\phi_4(z, s)) \right\} - \left\{ r \theta V(A_1(z, s)) \lambda_1 C[g(z_2)] + \left( 1 - \theta V(A_2(z, s)) + \theta V(A_1(z, s)) \right) \times \overline{M}(s + \lambda_1 + \lambda_2) + \lambda_2 p_2 C(z_2) \left[ \frac{1 - \overline{M}(s + \lambda_1 + \lambda_2)}{(s + \lambda_1 + \lambda_2)} \right] \right\} \times \left\{ B_2(\phi_4(z, s)) \left[ q_3 + z_2 p_3 \right] + \lambda_2 p_2 \left[ \frac{1 - B_2(\phi_4(z, s))}{\phi_4(z, s)} \right] \right\} \]

\[ T_{(0)}(0, z_2, s) = \frac{h_3}{g^2} \]  \hspace{1cm} (29)
where
\[
\begin{align*}
  h_3 &= -\left\{(s + \lambda_1 + \lambda_2)I_{0,0} - 1\right\}\left\{1 - \theta \overline{V}(A_2(z, s)) + \theta \overline{V}(A_1(z, s))\right\} z_2 \\
  &\quad + \left\{(s + \lambda_1 + \lambda_2)I_{0,0} - 1\right\} \\
  &\quad \times \left\{q_3 + z_2 p_3 + \lambda_2 p_2 C(z_2)\right\} r \theta \overline{V}(A_1(z, s)) \\
  &\quad \times \left\{\overline{B}_2(\psi_1(z, s)) - \overline{B}_2(\phi_4(z, s))\right\} + \lambda_2 p_2 C(z_2)I_{0,0} \\
  &\quad \times \left\{1 - \theta \overline{V}(A_2(z, s)) + \theta \overline{V}(A_1(z, s))\right\} \left\{\overline{B}_2(\phi_4(z, s))\right\} \left\{q_3 + z_2 p_3\right\} \\
  &\quad + \lambda_2 p_2 \left[1 - \frac{\overline{B}_2(\phi_4(z, s))}{\phi_4(z, s)}\right] \\
  &\quad + \lambda_2 p_2 C(z_2)I_{0,0} \left[q_3 + z_2 p_3 + \lambda_2 p_2 C(z_2)\right] \left[1 - \theta + (1 - r)\theta \overline{V}(A_1(z, s))\right] \\
  &\quad \times \left\{\overline{B}_2(\psi_1(z, s)) - \overline{B}_2(\phi_4(z, s))\right\}
\end{align*}
\]

\[
\overline{P}(0, z_1, z_2, s) = \frac{h_4}{g^3}
\] (30)

where
\[
\begin{align*}
  h_4 &= I_{(0)}(0, z_2, s) \left\{1 - A(z, s)\right\} + \overline{P_0}(0, z_2, s) \left[q_3 + z_2 p_3 + \lambda_2 p_2 C(z_2)\right] \\
  &\quad \times \left\{\overline{B}_2(\phi_2(z, s)) - \overline{B}_2(\psi_1(z, s))\right\}
\end{align*}
\]

\[
g^3 = z_1 - \left[q_1 + z_1 p_1\right] \overline{B}_1(\phi_1(z, s))
\]

other boundary conditions are
\[
\overline{V}_0(0, z_2, s) = \frac{\theta \left[\lambda_1 C[g(z_2)]I_{(0)}(0, z_2, s) + \overline{q_0}(0, z_2, s)\right] \left[q_3 + z_2 p_3 + \lambda_2 p_2 C(z_2)\right] \\
  &\quad \times \left\{\overline{B}_2(\psi_1(z, s)) - \overline{B}_2(\phi_4(z, s))\right\}}{\left\{1 - \theta \overline{V}(A_2(z, s)) + \theta \overline{V}(A_1(z, s))\right\}}
\] (31)

\[
\overline{D}_1(0, z_1, z_2, s) = a \overline{P}(0, z_1, z_2, s) \left[1 - \frac{\overline{B}_1(\phi_1(z, s))}{\phi_1(z, s)}\right]
\] (32)

\[
\overline{D}_2(0, z_1, z_2, s) = a \overline{q_0}(0, z_2, s) \left[1 - \frac{\overline{B}_2(\phi_2(z, s))}{\phi_2(z, s)}\right]
\] (33)
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\[
\bar{R}_1(0, z_1, z_2, s) = \alpha \bar{P}(0, z_1, z_2, s) \bar{D}_1(A(z, s)) \left[ \frac{1 - \bar{B}_1(\phi_1(z, s))}{\phi_1(z, s)} \right]
\]

(34)

\[
\bar{R}_2(0, z_1, z_2, s) = \alpha \bar{q}_0(0, z_2, s) \bar{D}_2(A(z, s)) \left[ \frac{1 - \bar{B}_2(\phi_2(z, s))}{\phi_2(z, s)} \right]
\]

(35)

However, by its definition \( \bar{q}(0, z_1, z_2, s) = \bar{q}_0(0, z_2, s) \) and \( \bar{V}(0, z_1, z_2, s) = \bar{V}_0(0, z_2, s) \).

**Theorem 4.1.** The inequality \( \rho_1 + \rho_2 < 1 \) is the necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server’s state, queue size and orbit size distributions are given by

\[
\bar{T}_{(0)}(z_2, s) = \bar{T}_{(0)}(0, z_2, s) \left[ \frac{1 - \bar{M}(s + \lambda_1 + \lambda_2)}{s + \lambda_1 + \lambda_2} \right],
\]

(36)

\[
\bar{P}(z_1, z_2, s) = \bar{P}(0, z_1, z_2, s) \left[ \frac{1 - \bar{B}_1(\phi_1(z, s))}{\phi_1(z, s)} \right],
\]

(37)

\[
\bar{q}(z_1, z_2, s) = \bar{q}_0(0, z_2, s) \left[ \frac{1 - \bar{B}_2(\phi_2(z, s))}{\phi_2(z, s)} \right],
\]

(38)

\[
\bar{V}(z_1, z_2, s) = \bar{V}_0(0, z_2, s) \left[ \frac{1 - \bar{V}(A(z, s))}{A(z, s)} \right],
\]

(39)

\[
\bar{D}_i(z_1, z_2, s) = \bar{D}_i(0, z_1, z_2, s) \left[ \frac{1 - \bar{D}_i(A(z, s))}{A(z, s)} \right],
\]

(40)

\[
\bar{R}_i(z_1, z_2, s) = \bar{R}_i(0, z_1, z_2, s) \left[ \frac{1 - \bar{R}_i(A(z, s))}{A(z, s)} \right].
\]

(41)

**Proof.** Integrating (22) to (25) with respect to \( x \) and using the well known result of renewal theory

\[
\int_0^\infty e^{-sx}(1 - F(x))dx = \frac{[1 - \bar{f}(s)]}{s}
\]

where \( F(x) \) is the distribution function of a random variable and \( \bar{f}(s) \) represents LST of it, we get the formulae (36) to (39), respectively. Similarly, integrating (26) and (27) with respect to \( y \), we get

\[
D_i(x, z) = \int_0^\infty D_i(x, y, z)dy = \frac{\alpha(\bar{P}(x, z) + \bar{q}(x, z))[1 - \bar{D}_i(b(z))]}{b(z)},
\]

(42)
\[ R_i(x, z) = \int_0^\infty R_i(x, y, z)dy = \frac{\alpha(\overline{P}(x, z) + \overline{q}(x, z))\overline{D}_i[b(z)][1 - \overline{R}_i[b(z)]]}{b(z)}. \] (43)

Further integrating (42) and (43) with respect to \( x \), we can get the formula (40) and (41), respectively. Thus we can obtain the complete solution for the probability generating function for the following states \( \overline{T}_0(z_2, s) \), \( \overline{P}(z_1, z_2, s) \), \( \overline{q}(z_1, z_2, s) \), \( \overline{V}(z_1, z_2, s) \), \( \overline{D}_i(z_1, z_2, s) \) and \( \overline{R}_i(z_1, z_2, s) \).

References


