Exact Traveling Wave Solutions of the special CH-DP Equation

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Abstract

In this paper, twelve exact explicit parametric representations of the traveling wave solutions for the special CH-DP equation are considered. By using the methods of planar dynamical systems, in different parameter regions, bifurcations of phase portraits of the corresponding singular traveling wave system are given.

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1. Introduction

It is well known that the Camassa-Holm equation

\[ u_t + 2k u_x - u_{xxt} + 3u u_x = 2u_x u_{xx} + uu_{xxx} \]  

was proposed by Camassa and Holm (Camassa et al., 1993) as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties, e.g., it is an integrable equation and admits a peakon solution. For \( k = 0 \), Camassa and Holm (Camassa et al., 1993) showed that (1) has peaked solitary wave solutions.

A new variant of (1) was introduced by Degasperis and Procesi (1999):

\[ u_t + c_0 u_x + bu_{xxx} - a^2 u_{xxx} = (c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx})_x \]  

where \( a, b \) and \( c_j (j = 0, 1, 2, 3) \) are constant parameters. In recent years, much work with the CH-DP equation has been done (Degasperis et al., 1999; Li et al., 2009; Liang
et al., 2009; Xie et al., 2010; Holm et al., 2003; Dullin et al., 2004). The special case $a = \pm 1$ and $b = c_0 = 0$, $c_1 = -2$, $c_2 = 3/2$, $c_3 = 1$, authors studied exact loop solutions, cusp solutions, solitary wave solutions and periodic wave solutions of (2) (Li et al., 2009), the special case $a = \pm 1$, $b = c_0 = 0$, $c_1 = -3/2$, $c_2 = c_3 = 1$ and $a = \pm 1$, $b = c_0 = 0$, $c_1 = -2$, $c_2 = 3/2$, $c_3 = 1$, some new traveling wave solutions of (2) have been given (Liang et al., 2009).

In this paper, we consider the special case $c_2 = 1/2$, $c_3 = 1$ and $a$, $b$, $c_0$, $c_1$ are constant parameters, then Eq. (2) becomes

$$u_t + c_0 u_x + b u_{xxx} - a^2 u_{txx} = \left( c_1 u^2 + \frac{1}{2} u_x^2 + uu_{xx} \right)_x$$  \hspace{1cm} (3)

Since the dynamical behavior of the corresponding traveling wave systems of (3) has not been studied before, in this paper, we shall consider the dynamical bifurcation (Li et al., 2007; Li et al., 2006; Li et al., 2007) of the traveling wave systems and obtain all possible exact explicit parametric representations of traveling wave solutions of (3). To investigate the traveling wave solutions of (3), let

$$u(x, t) = \phi(x - ct) = \phi(\xi), \hspace{1cm} \xi = x - ct$$  \hspace{1cm} (4)

Substituting (4) into (3) and integrating (3) once with respect to $\xi$, we obtain

$$(c_0 - c)\phi + (b + a^2 c) \phi \xi \xi = c_1 \phi^2 + \frac{1}{2} \phi^2_x + \phi \phi \xi \xi + g$$  \hspace{1cm} (5)

where $g$ is the integral constants.

Write $q = b + a^2 c$. Equation (5) is equivalent to the planar dynamical system

$$\frac{d\phi}{d\xi} = y, \hspace{1cm} \frac{dy}{d\xi} = \frac{1}{2} y^2 + c_1 \phi^2 - (c_0 - c)\phi + g$$  \hspace{1cm} (6)

Clearly, system (6) is a singular traveling wave system of the first type with the singular straight line $\phi = q$ (Li et al., 2007) and the first integral

$$H(\phi, y) = \frac{1}{2} y^2 (q - \phi) - \frac{c_1}{3} \phi^3 + \frac{c_0 - c}{2} \phi^2 - g\phi$$  \hspace{1cm} (7)

To investigate the dynamics of the orbits of (6), we first consider the associated regular system of (6) as follows:

$$\frac{d\phi}{d\zeta} = y(q - \phi), \hspace{1cm} \frac{dy}{d\zeta} = \frac{1}{2} y^2 + c_1 \phi^2 - (c_0 - c)\phi + g$$  \hspace{1cm} (8)

where $d\zeta = (q - \phi)d\xi$

Now, $\phi = q$ is an invariant straight line solution of (8). System (6) and (8) have the same invariant curve solutions defined by (7), but the dynamics and parametric representations of the same orbit are different (Li et al., 2009). We shall explain this phenomenon in section 2.
2. Bifurcations of phase portraits of system (6)

In this section, we first study the bifurcations of phase portraits of (6).

Define $\Delta = (c_0 - c)^2 - 4c_1 g$. Clearly, when $\Delta < 0$, there is no equilibrium point of (8) on the $\phi$-axis. When $\Delta = 0$, there is a double equilibrium point $E_{1,2} \left( \frac{c_0 - c}{2c_1}, 0 \right)$ of (8). When $\Delta > 0$, there are two equilibrium points $E_1(\phi_1,0)$ and $E_2(\phi_2,0)$ of (8), where $\phi_{1,2} = \frac{(c_0 - c) \pm \sqrt{\Delta}}{2c_1}$.

For the Hamiltonian $H$ defined by (7), Let

$$h_1 = H(\phi_1,0) = \frac{(c_0 - c)^3}{12c_1^2} + \frac{(c_0 - c)^2 \sqrt{\Delta}}{8c_1^2} - \frac{\Delta \sqrt{\Delta}}{24c_1^2} - \frac{g(c_0 - c) + \sqrt{\Delta}}{2c_1},$$

$$h_2 = H(\phi_2,0) = \frac{(c_0 - c)^3}{12c_1^2} - \frac{(c_0 - c)^2 \sqrt{\Delta}}{8c_1^2} + \frac{\Delta \sqrt{\Delta}}{24c_1^2} - \frac{g(c_0 - c) - \sqrt{\Delta}}{2c_1},$$

$$h_3 = H(q, \pm Y) = -\frac{c_1}{3} q^3 + \frac{c_0 - c}{2} q^2 - gq.$$

Relational expression $\phi = q = \phi_1$, $\phi = q = \phi_2$ lead to the following branch curve line: $L_1 : B = \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}$ and $L_2 : B = \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}$, where $B = b + a^2 c_0, A = c_0 - c$, whole parameters plane $(A, B)$ is divided into seven different areas by two branch curve line $L_1, L_2$ and straight line $A = \pm 2 \sqrt{c_1 g}$ (Fig. 1 (1-1) and Fig. 5 (5-1)).

By the qualitative analysis, we know that for $c_1 > 0, A > 2 \sqrt{c_1 g}$, the bifurcations of the phase portraits of (8) are shown in Fig. 1 (1-2)-(1-6).

![Fig.1 The Bifurcations of phase portraits of (8) for $c_1 > 0, A > 2 \sqrt{c_1 g}$](image)
For every fixed \(c_1 > 0\), \(A = \pm 2\sqrt{c_1 g}\), the bifurcations of the phase portraits of (8) are shown in Fig. 2 (2-1)-(2-6).

\[
(2-1) \ A = 2\sqrt{c_1 g}, \quad B > \left( a^2 + \frac{1}{2c_1} \right) A. \quad (2-2) \ A = 2\sqrt{c_1 g}, \quad B = \left( a^2 + \frac{1}{2c_1} \right) A. \quad (2-3) \ A = 2\sqrt{c_1 g}, \quad B < \left( a^2 + \frac{1}{2c_1} \right) A.
\]

\[
(2-4) \ A = -2\sqrt{c_1 g}, \quad B = \left( a^2 + \frac{1}{2c_1} \right) A. \quad (2-5) \ A = -2\sqrt{c_1 g}, \quad B = \left( a^2 + \frac{1}{2c_1} \right) A. \quad (2-6) \ A = -2\sqrt{c_1 g}, \quad B < \left( a^2 + \frac{1}{2c_1} \right) A.
\]

For every fixed \(c_1 > 0\), \(A < -2\sqrt{c_1 g}\), the bifurcations of the phase portraits of (8) are shown in Fig. 3 (3-1)-(3-5).

\[
(3-1) \ B > \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}. \quad (3-2) \ B = \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}. \quad (3-3) \ A = \sqrt{A^2 - 4c_1 g}. \quad (3-4) \ B = \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}. \quad (3-5) \ B < \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}.
\]

For every fixed \(c_1 < 0\), \(A > 2\sqrt{c_1 g}\), the bifurcations of the phase portraits of (8) are shown in Fig. 4 (4-1)-(4-6).

\[
(4-1) \ B > \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}. \quad (4-2) \ B = \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}. \quad (4-3) \ B < \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}. \quad (4-4) \ B < \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}. \quad (4-5) \ B < \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1}.
\]
In this section, we give the parametric representations for all bounded solutions defined by $H(\phi, y) = h$ of (6) in different parameter conditions.
We see from (7) that
\[ y^2 = \frac{2c_1 \phi^3 - (c_0 - c)\phi^2 + 2g\phi + 2h}{q - \phi} \]
(9)

By using the first equation of (6) we obtain
\[
\int \sqrt{\frac{q - \phi}{2c_1 \phi^3 - (c_0 - c)\phi^2 + 2g\phi + 2h}} d\phi = \xi
\]
(10)

It follows the parametric representations of the orbits defined by \( H(\phi, y) = h \).

1. The case \( A > 2\sqrt{c_1 g}, B > \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1 g}}{2c_1} \) (see Figs. 1 (1-2)).

In this case, for every \( h \in (h_2, h_1) \), (9) can be written as
\[
y^2 = \frac{2c_1 (r_1 - \phi)(r_2 - \phi)(\phi - r_3)}{q - \phi}
\]

Thus, we obtain a family of periodic solutions of (6) as follows:
\[
\phi = \frac{q(r_2 - r_3)sn^2(\omega, k) + r_3(q - r_2)}{(r_2 - r_3)sn^2(\omega, k) + (q - r_2)},
\]
\[
\xi = \sqrt{\frac{3}{2c_1}}(q - r_3)g_1 \Pi(arcsin(sn(\omega, k)), \alpha^2, k).
\]
(11)

where
\[
g_1 = \frac{2}{\sqrt{(q - r_2)(r_1 - r_3)}}, \alpha^2 = \frac{r_3 - r_2}{q - r_2}, k^2 = \frac{(q - r_1)(r_2 - r_3)}{(q - r_2)(r_1 - r_3)}.
\]

For \( h = h_1 \), (9) has the form \( y^2 = \frac{2c_1 (\phi_1 - \phi)^2(\phi - r)}{3 \frac{q - \phi}{q - \phi_1}} \), Hence, the first equation of (6) gives rise to the following solitary wave solution:
\[
\phi = \frac{q(\phi_1 - r)tanh^2\omega + r(q - \phi_1)}{(\phi_1 - r)tanh^2\omega + (q - \phi_1)},
\]
\[
\xi = \sqrt{\frac{3}{2c_1}}(q - r_3)g_1 \Pi(arcsin(tanh(\omega)), \alpha^2, 1).
\]
(12)

where
\[
g_1 = \frac{2}{\sqrt{(q - \phi_1)(\phi_1 - r)}}, \alpha^2 = \frac{r - \phi_1}{q - \phi_1}.
\]
2. The case \( A > 2\sqrt{c_1}g \), \( B = \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs. 1 (1-3)).

For \( h \in (h_2, h_1) \), the corresponding level curves defined by (7) determine respectively a family of periodic orbits (see Figs. 1 (1-3)). Now, (9) can be written as \( y^2 = \frac{2c_1}{3} \), \( q - \phi \), respectively. Thus, we obtain a parametric representation of a family of periodic wave solutions of (6) as follows:

\[
\phi = \frac{r_2(q - r_3) - q(r_2 - r_3)sn^2(\omega, k)}{(q - r_3) - (r_2 - r_3)sn^2(\omega, k)},
\]

\[
\xi = \sqrt{\frac{3}{2c_1}}(r_2 - q)g_1 \Pi(\arcsin(sn(\omega, k)), a^2, k).
\]  
(13)

where

\[
g_1 = \frac{2}{\sqrt{(q - r_3)(r_1 - r_2)}}, a^2 = \frac{r_2 - r_3}{q - r_3}, k^2 = \frac{(r_1 - q)(r_2 - r_3)}{(q - r_3)(r_1 - r_2)}.
\]

For \( h = h_1 \), we have \( q = \phi_1 \), (9) can be written as \( y^2 = \frac{2c_1}{3}(q - \phi)(\phi - r) \), by using the first equation of (6), we obtain the following periodic wave solutions:

\[
\phi(\xi) = \frac{q + r}{2} - \frac{q - r}{2} \cos(\sqrt{\frac{2c_1}{3}}\xi).
\]  
(14)

3. The case \( A > 2\sqrt{c_1}g \), \( \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1g}}{2c_1} < B < \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs. 1 (1-4)).

For \( h \in (h_1, h_2) \), the level curves defined by \( H(\phi, y) = h \) are a family of closed orbits of (6) in the right phase plane. Now, (9) can be written as \( y^2 = \frac{2c_1}{3}(r_1 - \phi)(\phi - r_2)(\phi - r_3) \).

By using the first equation of (6), we obtain a parametric representation of a family of periodic wave solutions of (6) as follows:

\[
\phi = \frac{r_2(r_1 - q) - q(r_1 - r_2)sn^2(\omega, k)}{(r_1 - q) - (r_1 - r_2)sn^2(\omega, k)},
\]

\[
\xi = \sqrt{\frac{3}{2c_1}}(r_2 - q)g_1 \Pi(\arcsin(sn(\omega, k)), a^2, k).
\]  
(15)

where

\[
g_1 = \frac{2}{\sqrt{(r_1 - q)(r_2 - r_3)}}, a^2 = \frac{r_1 - r_2}{r_1 - q}, k^2 = \frac{(r_1 - r_2)(q - r_3)}{(r_1 - q)(r_2 - r_3)}.
\]
For \( h \in (h_2, h_1) \), the level curves defined by \( H(\phi, y) = h \) are a family of closed orbits of (6) in the left phase plane. We obtain a parametric representation of a family of periodic wave solutions of (6) as (13).

For \( h = h_s \), (9) can be written as \( y^2 = \frac{2c_1}{3}(r_1 - \phi)(\phi - r_2) \), by using the first equation of (6), we obtain the following periodic wave solutions:

\[
\phi(\xi) = \frac{r_1 + r_2}{2} - \frac{r_1 - r_2}{2} \cos\left(\sqrt{\frac{2c_1}{3}} \xi\right).
\] (16)

\[ \text{4. The case } A > 2\sqrt{c_1 g}, \quad B = \left(\alpha^2 + \frac{1}{2c_1}\right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1} \] (see Figs. 1 (1-5)).

For \( h \in (h_2, h_1) \), the corresponding level curves defined by (7) determine respectively a family of periodic wave solutions of (6), which has the same parametric representations as (15).

For \( h = h_2 \), we have \( q = \phi_2 \), (9) can be written as \( y^2 = \frac{2c_1}{3}(r - \phi)(\phi - q) \), we obtain a parametric representation of a family of periodic wave solutions of (6) as (14).

\[ \text{5. The case } A > 2\sqrt{c_1 g}, \quad B < \left(\alpha^2 + \frac{1}{2c_1}\right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1} \] (see Figs. 1 (1-6)).

For \( h \in (h_2, h_1) \), (9) can be written as \( y^2 = \frac{2c_1}{3}(r_1 - \phi)(\phi - r_2)(\phi - r_3) \). Thus, we obtain a family of periodic solutions of (6) as follows:

\[
\phi = \frac{q(r_1 - r_2)sn^2(\omega, k) + r_1(r_2 - q)}{(r_1 - r_2)sn^2(\omega, k) + (r_2 - q)},
\]

\[
\xi = \sqrt{\frac{3}{2c_1}}(q - r_1)g_1 \Pi(\arcsin(sn(\omega, k)), \alpha^2, k).
\] (17)

where

\[
g_1 = \frac{2}{\sqrt{(r_2 - q)(r_1 - r_3)}}, \quad \alpha^2 = \frac{r_2 - r_1}{r_2 - q}, \quad k^2 = \frac{(r_1 - r_2)(r_3 - q) + (r_1 - r_3)(r_2 - q)}{(r_1 - r_3)(r_2 - q)}.
\]

For \( h = h_2 \), (9) has the form \( y^2 = \frac{2c_1}{3}(r - \phi)(\phi - \phi_2)^2 \), Hence, the first equation of (6) gives rise to the following solitary wave solution:

\[
\phi = \frac{q(r - \phi_2)tanh^2\omega + r(\phi_2 - q)}{(r - \phi_2)tanh^2\omega + (\phi_2 - q)},
\]

\[
\xi = \sqrt{\frac{3}{2c_1}}(q - r)g_1 \Pi(\arcsin(tanh\omega), \alpha^2, 1).
\] (18)
where
\[ g_1 = \frac{2}{\sqrt{(r - \phi_2)(\phi_2 - q)}}, \alpha^2 = \frac{\phi_2 - r}{\phi_2 - q}. \]

For \( h \in (h_2, h_1) \), the level curves defined by \( H(\phi, y) = h \) are a open orbit of (6) in the left phase plane, (9) can be written as \( y^2 = \frac{2c_1}{3} (r_1 - \phi)(r_2 - \phi)(r_3 - \phi) \). By using (6), corresponding to the open orbit, we have the parametric representation as follows:
\[ \phi = \frac{r_3(r_2 - q) - r_2(r_3 - q)sn^2(\omega, k)}{(r_2 - q) - (r_3 - q)sn^2(\omega, k)}, \]
\[ \xi = \sqrt{\frac{3}{2c_1}} g_1 [(r_2 - q)\omega + (r_3 - r_2)\Pi(arcsin(sn(\omega, k)), \alpha^2, k)]. \] (19)

where
\[ g_1 = \frac{2}{\sqrt{(r_1 - r_3)(r_2 - q)}}, \alpha^2 = \frac{r_3 - q}{r_2 - q}, k^2 = \frac{(r_1 - r_2)(r_3 - q)}{(r_1 - r_3)(r_2 - q)}. \]

6. The case \( A < -2\sqrt{c_1g}, B > (a^2 + \frac{1}{2c_1}) A + \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs.3 (3-1)).

For \( h \in (h_2, h_1) \), the corresponding level curves defined by (7) determine respectively a family of periodic wave solutions of (6), which has the same parametric representations as (11).

For \( h = h_1 \), the corresponding level curves defined by (7) determine respectively the solitary wave solutions of (6), which has the same parametric representations as (12).

7. The case \( A < -2\sqrt{c_1g}, B = (a^2 + \frac{1}{2c_1}) A + \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs.3 (3-2)).

For \( h \in (h_2, h_1) \), the corresponding level curves defined by (7) determine respectively a family of periodic wave solutions of (6), which has the same parametric representations as (13).

For \( h = h_1 \), the corresponding level curves defined by (7) determine respectively the periodic wave solutions of (6), which has the same parametric representations as (14).

8. The case \( A < -2\sqrt{c_1g}, (a^2 + \frac{1}{2c_1}) A - \frac{\sqrt{A^2 - 4c_1g}}{2c_1} < B < (a^2 + \frac{1}{2c_1}) A + \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs.3 (3-3)).

For \( h \in (h_s, h_1) \), the level curves defined by \( H(\phi, y) = h \) are a family of closed orbits of (6) in the right phase plane. We obtain a parametric representation of a family of periodic wave solutions of (6) as (15).
For \( h \in (h_2, h_3) \), the level curves defined by \( H(\phi, y) = h \) are a family of closed orbits of (6) in the left phase plane. We obtain a parametric representation of a family of periodic wave solutions of (6) as (13).

For \( h = h_s \), the corresponding level curves defined by (7) determine respectively the periodic wave solutions of (6), which has the same parametric representations as (16).

9. The case \( A < -2\sqrt{c_1 g}, B = \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1} \) (see Figs. 3 (3-4)).

For \( h \in (h_2, h_1) \), the corresponding level curves defined by (7) determine respectively a family of periodic wave solutions of (6), which has the same parametric representations as (15).

For \( h = h_2 \), the corresponding level curves defined by (7) determine respectively the periodic wave solutions of (6), which has the same parametric representations as (14).

10. The case \( A < -2\sqrt{c_1 g}, B < \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1} \) (see Figs. 3 (3-5)).

For \( h \in (h_2, h_1) \), the corresponding level curves defined by (7) determine respectively a family of periodic wave solutions of (6), which has the same parametric representations as (15).

For \( h = h_2 \), the corresponding level curves defined by (7) determine respectively the solitary wave solutions of (6), which has the same parametric representations as (18).

4. The parametric representations of bounded orbits defined by \( H(\phi, y) = h \) of (6) when \( c_1 < 0 \)

1. The case \( A > 2\sqrt{c_1 g}, B > \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1 g}}{2c_1} \) (see Figs.4 (4-2)).

In this case, for every \( h \in (h_1, h_2) \), (9) can be written as

\[
y^2 = \frac{2|c_1|}{3} \left( \frac{r_1 - \phi}{q - \phi} - (q - r_2)(\phi - r_3) \right)
\]

Thus, we obtain a family of periodic solutions of (6) as follows:

\[
\phi = \frac{r_1(q - r_2) - q(r_1 - r_2)sn^2(\omega, k)}{(q - r_2) - (r_1 - r_2)sn^2(\omega, k)},
\]

\[
\xi = \frac{3}{2|c_1|}(r_1 - q)g_1 \Pi(arcsin(sn(\omega, k)), a^2, k).
\]

where

\[
g_1 = \frac{2}{\sqrt{(q - r_2)(r_1 - r_3)}}, a^2 = \frac{r_1 - r_2}{q - r_2}, k^2 = \frac{(q - r_3)(r_1 - r_2)}{(q - r_2)(r_1 - r_3)}.
\]
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For \( h = h_2 \), (9) has the form \( y^2 = \frac{2|c_1|(r - \phi)(\phi - \phi_2)^2}{3(q - \phi)} \), Hence, the first equation of (6) gives rise to the following solitary wave solution:

\[
\phi = \frac{r(q - \phi_2) - q(r - \phi_2)\tanh^2 \omega}{(q - \phi) - (r - \phi_2)\tanh^2 \omega},
\]

\[
\xi = \sqrt{\frac{3}{2|c_1|}}(r - q)g_1 \Pi(\arcsin(\tanh \omega), \alpha^2, 1). \tag{21}
\]

where \( g_1 = \frac{2}{\sqrt{(q - \phi)(r - \phi_2)}}, \alpha^2 = \frac{r - \phi_2}{q - \phi_2}. \)

2. The case \( A > 2\sqrt{c_1g}, B < \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs. 4 (4-6)).

For \( h \in (h_2, h_2) \), (9) can be written as

\[
y^2 = \frac{2|c_1|(r_1 - \phi)(r_2 - \phi)(\phi - r_3)}{3(q - \phi)}
\]

Thus, we obtain a family of periodic solutions of (6) as follows:

\[
\phi = \frac{r_3(r_2 - q) - q(r_2 - r_3)\text{sn}^2(\omega, k)}{(r_2 - q) - (r_2 - r_3)\text{sn}^2(\omega, k)},
\]

\[
\xi = \sqrt{\frac{3}{2|c_1|}}(r_3 - q)g_1 \Pi(\arcsin(\text{sn}(\omega, k)), \alpha^2, k). \tag{22}
\]

where \( g_1 = \frac{2}{\sqrt{(r_2 - q)(r_1 - r_3)}}, \alpha^2 = \frac{r_2 - r_3}{r_2 - q}, k^2 = \frac{(r_2 - r_3)(r_1 - q)}{(r_1 - r_3)(r_2 - q)}. \)

3. The case \( A < -2\sqrt{c_1g}, B > \left( a^2 + \frac{1}{2c_1} \right) A - \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs. 6 (6-1)).

For \( h \in (h_1, h_3) \), the corresponding level curves defined by (7) determine respectively a family of periodic wave solutions of (6), which has the same parametric representations as (20).

4. The case \( A < -2\sqrt{c_1g}, B < \left( a^2 + \frac{1}{2c_1} \right) A + \frac{\sqrt{A^2 - 4c_1g}}{2c_1} \) (see Figs. 6 (6-5)).

For \( h \in (h_1, h_2) \), the corresponding level curves defined by (7) determine respectively a family of periodic wave solutions of (6), which has the same parametric representations as (22).
To sum up, for $c_1 > 0$ and $c_1 < 0$, under different parameter conditions, we obtain twelve exact parametric representations of $\phi(\xi)$ given by (11)-(22).

References


