International Journal of Pure and Applied Physics
ISSN 0973-1776 Volume 4, Number 3 (2008), pp. 233–250

http://www.ripublication.com/ijpap.htm

Exactly Solvable Schrödinger Equation for Multiterm Quantum Mechanical Potentials

S.A.S. Ahmed[1*], L. Buragohain[2] and N. Saikia[2]

[1]Department of Physics, Gauhati University, Guwahati-781014, India
E-mail:sasa_gup@hotmail.com
[2]Department of Physics, Chaiduar College, Gohpur-784168, India

Abstract

Exact bound state solutions of Schrödinger equation for various new multiterm potentials are obtained in any chosen dimensional space, using Extended transformation (ET) method which may find applications in Atomic, Molecular, Nuclear and Particle Physics. We have found for multiterm powerlaw potentials, under the framework of ET, that a family relationship emerges among the parent and the newly generated exactly solvable potentials (ESPs). For a k-term powerlaw potential, repeated applications of ET generates k different ESPs. Choosing any one term out of k+1 potentials and performing ET we get, in general, no further new potentials but another among the multiplet of other generated k -ESPs. The normalizability of bound state solutions of the generated ESPs are discussed.

Key words: Exactly solvable potential; Dual and self-dual quantum potential; Schrödinger equation; Transformation method.

PACS number(s): 02.90.+p, 03.65.Db, 03.65.Ge

Introduction

Exact analytic solution (EAS) of Schrödinger equation for a given physical quantum system (QS) is desirable as it conveys maximum information of the system. Considerable efforts have been made in recent years towards obtaining exact solution of the Schrödinger equation for potentials of physical interest [1-6]. However only a very few potentials governing physical systems yield to analytical solutions. Even to perform an approximate calculation in an efficient way one needs an exactly solvable potential (ESP) “near by”. This prompted us to explore and find as many exactly solvable potentials (ESPs) that may exist and which may find applications in different branches of Physics and Chemistry. For this purpose we have used the extended transformation (ET) method [7] to generate new ESPs in any preassigned dimensional space. In this paper we confined ourselves to multiterm potentials only. The extended transformation (ET) includes a coordinate transformation (CT) followed by a functional transformation (FT).

Actually CT alone can generate new ESPs from an old one. But it leads to perform regarding dimensionality of the Euclidean space in to which the transformed quantum system (QS) gets transported to. The difficulty can be overcome by supplementing the CT by FT on the original exactly solvable quantum system. The FT allows us to have dimensional extension and/or dimensional reduction of the generated quantum systems. We obtain here a class of new exactly solved potentials and EAS of the Schrödinger equation for some potential which includes two and three term fractional power potentials and singular integer power potential. Starting from an exactly solved k-term potential, ET can in principle generate 2^{k-1} different new exactly solved potentials. A very useful property of the ET method is that the wavefunctions of the generated quantum systems (QSs) are almost always normalizable provided the behaviour of a certain transformation function $g_B(r)$is smooth.

The paper is organized as follows: In section 2 formalism of ET is given briefly. In subsections (3.1-3.3) new ESPs are generated from already known two and three term fractional power potentials and singular integral power potential which are shown in Tables (1-3). Family relationship is shown in Figures (4-6) for generated two term and three term fractional power potentials and integral power singular self-dual potential. Section 4 comprises of the conclusion of our investigation.

Formalism

General Central Potential

The extended transformation method (ET) [7] has been applied to generate new exactly solved potentials (ESPs) from an already known ESP.

Let $V_A(r)$ is an exactly solved multiterm quantum mechanical central potential in D_A-dimensional space of the form given by:

$$V_A(r)=\alpha_1 r^{a_1}+\alpha_2 r^{a_2}+\alpha_3 r^{a_3} \tag{1}$$

Where α_1, α_2 and α_3 are the parameters of the potential and the QS will be called A-QS.

The radial part of reduced Schrödinger equation for D_A- dimensional Euclidean space for A-QS $(\hbar=2m=1)$ is:

$$\Psi_A''(r)+\frac{D_A-1}{r}\Psi_A'(r)+\left[E_A-V_A(r)-\frac{l_A(l_A+D_A-2)}{r^2}\right]\Psi_A(r)=0 \tag{2}$$

Where the normalized eigenfunctions $\Psi_A(r)$ and energy eigenvalue E_A are known for the given $V_A(r)$. Prime denotes the differentiation of the function with respect to its argument.

Energy eigenvalue of the A-QS is given by the function

$$E_A = F(\alpha_1, \alpha_2, \alpha_3; q)$$

When the potential has two or more terms, exact solvability is found under certain constraint equation:

$$F_A(\alpha_1, \alpha_2, \alpha_3; q) = 0$$

Where ‘q’ is a composite quantum number.

Under extended transformation (ET), which consists of a coordinate transformation

$r \to g_B(r)$, followed by a functional transformation of the wave function:

$$\Psi_B(r) = f^{-1}(r)\Psi_A(g_B(r)) \tag{3}$$

Where the transformation function $g_B(r)$ and the modulated amplitude function $f_B(r)$ have to be specified within the framework of ET.

As $\Psi_A(r)$ is the eigenfunction of an exactly solved potential, hence $\Psi_B(r)$ gets specified exactly, henceforth called B-QS.

The transformed B-QS after implementing ET on A-QS become:

$$\begin{aligned} &\Psi_B''(r) + \left(\frac{d}{dr}\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'}\right)\Psi_B'(r) + \\ &\left[\left(\frac{d}{dr}\ln f_B\right)\left(\frac{d}{dr}\ln\frac{f_B' g_B^{D_A-1}}{g_B'}\right) + g_B'^2\left(E_A - V_A(g_B(r)) - \frac{l(l_A + D_A - 2)}{g_B^2}\right)\right]\Psi_B(r) = 0. \end{aligned} \tag{4}$$

To mould the above equation to a Schrödinger equation form in a chosen

D_B-dimensional Euclidean space, we consider the co-efficient of $\Psi_B'(r)$ in Eq. (4) as:

$$\frac{d}{dr}\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'} = \frac{d}{dr}\ln r^{D_B-1} \tag{5}$$

Integrating,

$$\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'} = \ln r^{D_B-1} - 2\ln N\,. \tag{6}$$

This gives:

$$f_B(r) = N g_B'^{\frac{1}{2}} g_B^{-\frac{D_A-1}{2}} r^{\frac{D_B-1}{2}}\,. \tag{7}$$

The transformation function $g_B(r)$ is at least three times differentiable and $f_B(r)$ is non-singular function of r.

The corresponding D_B-dimensional standard Schrödinger equation for the B-QS is found to be:

$$\Psi_B''(r)+\frac{D_B-1}{r}\Psi_B'(r)+ \\ \left[\frac{1}{2}\{g_B,r\}-\frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g_B'}{g_B}\right)^2+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\right. \\ \left.+g_B'^2\left(E_A-V_B\left(g_B(r)\right)-\frac{\left(l_A+\frac{D_A}{2}-1\right)^2}{g_B^2}+\frac{(D_A-2)^2}{4g_B^2}\right)\right]\Psi_B(r)=0 \tag{8}$$

Where

$$V_A\left(g_B(r)\right)=\alpha_1 g_B{}^{a_1}+\alpha_2 g_B{}^{a_2}+\alpha_3 g_B{}^{a_3} \tag{9}$$

And

$$\{g_B,r\}=\frac{g_B'''(r)}{g_B'(r)}-\frac{3}{2}\frac{g_B'''^2(r)}{g_B'^2(r)} \tag{10}$$

is the Schwartzian derivative symbol [15].

To implement ET on A-QS, we have to select a term of $V_A\left(g_B(r)\right)$ as a working potential $V_A^W\left(g_B(r)\right)$ and make the following ansatze:

$$g_B'^2(r)V_A{}^w\left(g_B(r)\right)=-E_B, \tag{11}$$

$$g_B'^2(r)E_A=-V_B^1(r), \tag{12}$$

$$-g_B'^2\left(V_A(g_B)-V_A^W(g_B)\right)=-V_B^2(r)\ , \tag{13}$$

$$\frac{g_B'^2\left(l_A+\frac{D_A}{2}-1\right)^2}{g_B^2}=\frac{\left(l_B+\frac{D_B}{2}-1\right)^2}{r^2}\ . \tag{14}$$

Let the working potential be:

$$V_A^W\left(g_B(r)\right)=\alpha_1 g_B^{a_1}(r) \tag{15}$$

which will specify, by Eq.(11), the functional form of the transformation function $g_B(r)$ and is:

$$g_B(r)=\left[\frac{a_1+2}{2}\left(-\frac{E_B}{\alpha_1}\right)r\right]^{\frac{2}{a_1+2}} \tag{16}$$

Obtained through a simple integration. The transformation function $g_B(r)$ has the desirable local property $g_B(0)=0$ by putting the integration constant equal to zero.

Eqs. (12) and (15) lead to:

$$V_B^1(r) = -E_A\left(\frac{a_1+2}{2}\left(-\frac{E_B}{\alpha_1}\right)^{\frac{1}{2}}\right)^{\frac{4}{a_1+2}}\left(\frac{2}{a_1+2}\right)^2 r^{-\frac{2a_1}{a_1+2}}$$

$$V_B^1(r) = C_B^2 r^{-\frac{2a_1}{a_1+2}} \;. \tag{17}$$

Where C_B^2 is the Characteristic Constant of B-QS and is:

$$C_B^2 = -E_A\left(\frac{a_1+2}{2}\left(-\frac{E_B}{\alpha_1}\right)^{\frac{1}{2}}\right)^{\frac{4}{a_1+2}}\left(\frac{2}{a_1+2}\right)^2 . \tag{18}$$

Eqs. (13) and (15) lead to:

$$V_B^2(r) = \beta_2 r^{\frac{-2a_1+2a_3}{a_1+2}} + \beta_3 r^{\frac{-2a_1+2a_2}{a_1+2}} \;. \tag{19}$$

Therefore the Potential of the generated B-QS becomes

$$V_B(r) = \beta_1 r^{b_1} + \beta_2 r^{b_2} + \beta_3 r^{b_3} \tag{20}$$

With the following exponents of the potential

$$b_1 = \frac{-2a_1}{a_1+2},\; b_2 = \frac{-2a_1+2a_3}{a_1+2} \;,\; b_3 = \frac{-2a_1+2a_2}{a_1+2} \tag{21}$$

And the following parameters of the potential:

$$\beta_1 = C_B{}^2,\; \beta_2 = \left(\frac{a_1+2}{2}\sigma\right)^{\frac{4+2a_3}{a_1+2}}\left(\frac{2}{a_1+2}\right)^2\alpha_3,$$

$$\beta_3 = \left(\frac{a_1+2}{2}\sigma\right)^{\frac{4+2a_3}{a_1+2}}\left(\frac{2}{a_1+2}\right)^2\alpha_2 \;. \tag{22}$$

The energy eigenvalue of B-QS is obtained from Eq. (18) and is:

$$E_B = -\alpha_1\left[\frac{C_B{}^2}{(-E_A)\left(\frac{a_1+2}{2}\right)^{-\frac{2a_1}{a_1+2}}}\right]^{\frac{a_1+2}{2}} . \tag{23}$$

The D_B-dimensional Schrödinger equation for B-QS reduces to:

$$\Psi_B''(r) + \frac{D-1}{r}\Psi_B'(r) + \left[E_B - V_B(r) - \frac{l_B(l_B+D_B-2)}{r^2}\right]\Psi_B(r) = 0 \;. \tag{24}$$

Generation of new exactly solved potentials

Two term fractional power singular potential

To generate new exactly solved potential we have applied our formalism on an exactly solved two term fractional power central potential [9-10]:

$$V_A(r) = \alpha_1 r^{-\frac{1}{2}} + \alpha_2 r^{-\frac{3}{2}} \tag{25}$$

Where the parameters are connected by the constraint equation:

$$\left(l_A + \frac{3}{4}\right)\{8\alpha_1(l_A+1)\}^{\frac{1}{3}} + \alpha_2 = 0 \tag{26}$$

Bose [9] had given the EAS of the QS as:

$$\Psi_A(r) = N_A r^{l_A} \exp\left[-(-E_A)^{\frac{1}{2}} r - \frac{\alpha_1}{(-E_A)^{\frac{1}{2}}} r^{\frac{1}{2}}\right] \tag{27}$$

The corresponding energy eigenvalue of A-QS are:

$$E_A = -\left[\frac{\alpha_1{}^2}{8(l_A+1)}\right]^{\frac{2}{3}}. \tag{28}$$

Selecting $\alpha_2 r^{-\frac{3}{2}}$ as the working potential (WP) and utilizing Eq. (16), we obtain

$$g_B(r) = \left(-\frac{E_B}{16\alpha_2}\right)^2 r^4. \tag{29}$$

This possesses the desirable local property $g_B(0) = 0$ and asymptotic property $g_B(\infty) = \infty$.

From Eqs. (12) and (13) we get the following potential of B-QS:

$$V_B(r) = \beta_{11} r^6 + \beta_{12} r^4 \tag{30}$$

With

$$\beta_{11} = \left[-E_A\left(-\frac{E_B}{8\alpha_2}\right)^4\right] = C_B{}^2 \tag{31}$$

And

$$\beta_{12} = 2\alpha_1\left[-\frac{E_B}{8\alpha_2}\right]^3 \tag{32}$$

Where $C_B{}^2$ is the Characteristic constant of B-QS.

The parameters β_{11} and β_{12} are related by the constraint equation:

$$\beta_{12} = 2\beta_{11}{}^{\frac{3}{4}}(2l_B + D + 2)^{\frac{1}{2}}. \tag{33}$$

Eq. (31) gives the energy eigenvalue of B-QS:

$$E_B = \frac{\beta_{12}}{2\beta_{11}{}^{\frac{1}{2}}}(2l_B + D). \tag{34}$$

The angular momentum quantum number l_B of B-QS is related to angular momentum quantum number l_A of A-QS through Eq. (14) and is:

$$4l_A+3=\left(l_B+\frac{D}{2}\right). \tag{35}$$

The Corresponding energy eigenvalue of B-QS now can be obtained from Eq. (3) and is:

$$\Psi_B(r)=N_B r^{l_B}\exp\left[-\frac{1}{4}\beta_{11}^{\frac{1}{2}}r^4-\frac{1}{2}\beta_{11}^{\frac{1}{4}}\left(2l_B+D+2\right)^{\frac{1}{2}}r^2\right]. \tag{36}$$

The Schrödinger equation for B-QS $(D_B=D)$ takes the following form:

$$\Psi_B''(r)+\frac{D-1}{r}\Psi_B'(r)+\left[E_B-\left(\beta_{11}r^6+\beta_{12}r^4\right)-\frac{l_B\left(l_B+D-2\right)}{r^2}\right]\Psi_B(r)=0. \tag{37}$$

The behaviour of the potential and some of the EAS of the B-QSs are given in Fig.1.

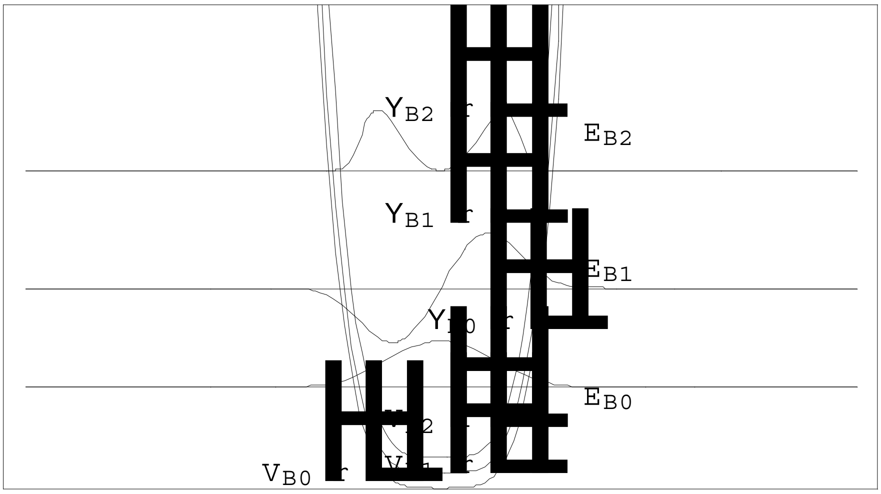

Figure 1: Labelled curves[1] for three different EQSs $\{E_{B0},V_{B0}(r),\Psi_{B0}(r)\}$, $\{E_{B1},V_{B1}(r),\Psi_{B1}(r)\}and\{E_{B0},V_{B0}(r),\Psi_{B0}(r)\}$ where the parameters sets are ($E_{B0}=6.7$, $\beta_{11}=1$, $\beta_{12}=4.47$, $l_B=0$, $D=3$). ($E_{B1}=13.22$, $\beta_{11}=1$, $\beta_{12}=5.29$, $l_B=1$, $D=3$). ($E_{B2}=21$, $\beta_{11}=1$, $\beta_{12}=6$, $l_B=2$, $D=3$).The graphs are drawn in arbitrary scale.

Considering other two term as working potential from $V_A(r)$ we can generate another ESPs which are shown in Table 1.

We have obtained following relations from Eq. (21)

$$A_1B_1=4,\ \ A_1B_2=2A_2$$

Writing $A_i=a_i+2$ and $B_j=b_j+2$.

---

[1] The apparent tunneling of the eigenfunctions are due to the arbitrary scale of energy levels and potential curves, and is not real.

These specifies the $B_J s$ of the dual B-QS from the known $A_j s$ and are the generalization of the duality relation of the one term potentials.

A further application of ET on the daughter QSs doesn't produce any new exactly solved quantum system (EQS) but depending on the WPs give one or the other EQS, bringing them in to a family of three EQSs.

Table 1: List of generated exactly solved potentials (ESPs) in D-dimensional Euclidean spaces generated from an already (A-QS) two term fractional power potential $V_A(r) = \alpha_1 r^{-\frac{1}{2}} + \alpha_2 r^{-\frac{3}{2}}$.

WP	**Exactly solved potential**	**Energy Eigenvalue**	**Constraint Equation**	**Energy EigenFunction**
$\alpha_2 g_B^{-\frac{3}{2}}$	$V_B(r) = \beta_{11} r^6 + \beta_{12} r^4$; $\beta_{11} = \left[-E_A \left(-\frac{E_B}{8\alpha_2} \right)^4 \right]$ $\beta_{12} = 2\alpha_1 \left[-\frac{E_B}{8\alpha_2} \right]^3$	$E_B = \frac{\beta_{12}}{2\sqrt{\beta_{11}}} (2l_B + D)$	$\beta_{12} = 2\beta_{11}^{\frac{3}{4}}$ $(2l_B + D + 2)^{\frac{1}{2}}$	$\Psi_B(r) = N_B r^{l_B} \exp$ $\left[-\frac{\sqrt{\beta_{11}}}{4} r^4 - \frac{\beta_{11}^{\frac{1}{4}}}{2} (2l_B + D + 2)^{\frac{1}{2}} r^2 \right]$
$\alpha_1 g_C^{-\frac{1}{2}}$	$V_C(r) = \beta_{21} r^{\frac{2}{3}} + \beta_{22} r^{-\frac{4}{3}}$; $\beta_{21} = \left(-\frac{\sqrt{3}E_C}{2\alpha_1} \right)^{\frac{4}{3}} (-E_A)$ $\beta_{22} = \alpha_2 \left(-\frac{4^4 E_C}{3^4 \alpha_1} \right)^{\frac{1}{3}}$	$E_C = -\frac{2}{\sqrt{3}} \beta_{21} r^{\frac{3}{4}}$ $(6l_C + 3D - 2)^{\frac{1}{2}}$	$\left(3\sqrt{3}\beta_{22}\right)^3 +$ $\beta_{21}^{\frac{3}{4}} (6l_C + 3D - 2)^{\frac{3}{2}}$ $(6l_C + 3D - 4)^3 = 0$	$\Psi_C(r) = N_C r^{l_C} \exp$ $\left[-\left(\frac{3}{4}\right)^{\frac{4}{3}} \frac{(-E_C)^{\frac{2}{3}}}{(6l_C + 3D - 2)^{\frac{1}{3}}} r^{\frac{4}{3}} - \right.$ $\left. -\left(\frac{3}{4} r\right)^{\frac{2}{3}} \{-E_C (6l_C + 3D - 2)\}^{\frac{1}{3}} \right]$

Three term fractional power potential:

In order to generate new exactly solved potential and also to investigate the family structure we consider a three term fractional power potential. Transformation Method [7] has been used to generate new EAS as a basis an already solved quantum potential [9]:

$$V_A(r)=\alpha_1 r^{\frac{2}{3}}+\alpha_2 r^{-\frac{2}{3}}+\alpha_3 r^{-\frac{4}{3}} \tag{38}$$

Where α_1, α_2 and α_3 are the parameters of the potential which are connected by the constraint equation as:

$$\left(2l_A+\frac{5}{3}\right)\left\{\left(2l_A+\frac{7}{3}\right)\sqrt{\alpha_1}+\alpha_2\right\}^{\frac{1}{2}}+\alpha_3=0\,. \tag{39}$$

The EAS of A-QS is

$$\Psi_A=N_A r^{l_A}\exp\left[-\frac{3}{4}\sqrt{\alpha_1}\,r^{\frac{4}{3}}+\frac{3}{4}\frac{E_A}{\sqrt{\alpha_1}}r^{\frac{2}{3}}\right], \tag{40}$$

With the energy eigenvalues

$$E_A=\pm\sqrt{4\alpha_1}\left\{\left(2l_A+\frac{7}{3}\right)\sqrt{\alpha_1}+\alpha_2\right\}^{\frac{1}{2}}.\ . \tag{41}$$

Selecting $\alpha_1 g_B^{\frac{2}{3}}(r)$ as the working potential and utilizing Eq. (16) we obtain

$$g_B(r)=\pm\left(\frac{16(-E_B)}{9a}\right)^{\frac{3}{8}} r^{\frac{3}{4}}+C, \tag{42}$$

Where C=0, implies that transformation function $g_B(r)$ has the desirable local property $g_B(0)=0$.

For power law $g_B(r)$, the following expression occurring in eq. (8)

$$\left[-\frac{1}{2}\{g_B,r\}+\frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g'_B}{g_B}\right)^2-\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\right.$$
$$\left.+\frac{g_B'^2}{g_B^{\,2}}\left(l_A+\frac{D_A}{2}-1\right)^2-\left(\frac{D_A-2}{2}\right)^2\left(\frac{g'_B}{g_B}\right)^2\right]$$

Becomes $\dfrac{l_B(l_B+D-2)}{r^2}$ by Eq. (14), considering $D_B=D$.

In Eq. (42) we have discarded the negative sign on the grounds that we are seeking bound states of a real potential. Once $g_B(r)$ become:

$$V_B(r)=\beta_{11}r^{-\frac{1}{2}}+\beta_{12}r^{-\frac{3}{2}}+\beta_{13}r^{-1}. \tag{43}$$

The coefficients β_{11}, β_{12} and β_{13} required for complete specification of $V_B(r)$ are given by:

$$\beta_{11}=\frac{\sqrt{3}}{2}\left(\frac{-E_B}{\alpha_1}\right)^{\frac{3}{4}}(-E_A) \tag{44}$$

$$\beta_{12}=\frac{3\sqrt{3}}{8}\alpha_3\left(\frac{-E_B}{\alpha_1}\right)^{\frac{1}{4}} \tag{45}$$

$$\beta_{13}=\frac{3}{4}\alpha_2\left(\frac{-E_B}{\alpha_1}\right)^{\frac{1}{2}}. \tag{46}$$

The parameters are connected by the constraint equation

$$\beta_{11}{}^{2}=4\beta_{13}\left[\frac{\beta_{11}}{4\beta_{12}}(4l_B+2D-3)\right]^2+4(2l_B+D-1)\left[\frac{\beta_{11}}{4\beta_{12}}(4l_B+2D-3)\right]^3. \tag{47}$$

Eqs. (14) and (42) yields the relationship between the angular momentum quantum numbers l_A and l_B of A-QS and B-QS as:

$$2l_A+1=\frac{4}{3}(2l_B+D-2). \tag{48}$$

The Schrödinger equation for B-QS of eq. (8) thus takes the form $(D_B=D)$:

$$\Psi_B''(r)+\frac{D-1}{r}\Psi_B'(r)+\left[E_B-\left(\beta_{11}r^{-\frac{1}{2}}+\beta_{12}r^{-\frac{3}{2}}+\beta_{13}r^{-1}\right)-\frac{l_B(l_B+D-2)}{r^2}\right]\Psi_B(r)=0$$

. (49)

Its EAS is given eq. (3) and is:

$$\Psi_B(r)=N_B r^{l_B}\exp\left[-\left(-E_B^{\frac{1}{2}}\right)r-\frac{\beta_{11}}{(-E_B)^{\frac{1}{2}}}r^{\frac{1}{2}}\right]. \tag{50}$$

From Eqs. (44) and (45) the energy eigenvalue spectrum of B-QS comes out to be:

$$E_B=-\left[\frac{\beta_{11}}{4\beta_{12}}(4l_B+2D-3)\right]^2. \tag{51}$$

Where the normalization constant is

$$N_B = \left[\frac{(-E_B)}{a < ar^{\frac{2}{3}} >_A} \right]^{\frac{1}{2}},$$

Which necessarily exists as the denominator is the expectation value of a part of the potential.

The behaviour of the potential and some of the EAS of the B-QS are given in Fig.2.

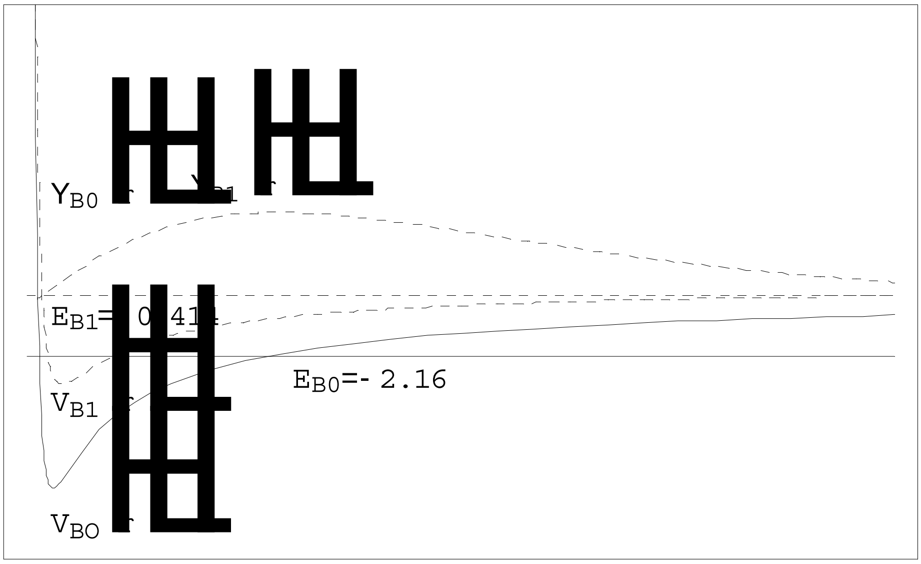

Figure 2: Continuous curves are for the EQS {E_{BO},V_{BO}(r),Ψ_{B0}(r)}, where the parameter set is (E_{B0}=-2.16, β_{11}=-1.96, β_{12}=1.0 , β_{13}=-2.5, l_C=0, D=3) and the broken curves are for the EQS {E_{B1}, V_{B1}(r), Ψ_{B1}(r)}, where the parameter set is (E_{B1}=-0.414, β_{11}=-0.368, β_{12}=1.0, β_{13}=-2.5, l_B=1, D=3). The graphs are drawn in arbitrary scale.

Like wise choosing other terms of the potential as the working potential we can generate another two exactly solved daughter quantum systems (QSs) having two different potentials. Preceding similarly other EQSs can be had. These are given in Table 2.

We have obtained following relations from Eq. (21):

$$A_1B_1 = 4 \ , \ A_1B_2 = 2A_3 \ \ A_1B_3 = 2A_2 \qquad (52)$$

Where $A_i = a_i + 2$ and $B_j = b_j + 2$.

These Eqs. Specifies the B_js of the dual B-QS from the known $A_j s$ of A-QS and the generalization of the duality relation of the one term potentials.

Table 2: List of ESPs in D-dimensional Euclidean spaces generated from an already known (A-QS) three term fractional power potential $\boldsymbol{V_A(r)=\alpha_1 r^{\frac{2}{3}}+\alpha_2 r^{-\frac{2}{3}}+\alpha_3 r^{-\frac{4}{3}}}$.

WP	Exactly solved potential	Energy Eigenvalue	Constraint Equation	Energy Eigen Function
$\alpha_1 g_B^{\frac{2}{3}}$	$V_B(r)=\beta_{11}r^{-\frac{1}{2}}+\beta_{12}r^{-\frac{3}{2}}+\beta_{13}r^{-1}$;	$E_B=-\left[\frac{\beta_{11}}{4\beta_{12}}(4l_B+2D-3)\right]^2$	$\beta_{11}^{\ 2}=4\beta_{13}\left[\frac{\beta_{11}}{4\beta_{12}}(4l_B+2D-3)\right]^2$ $+4(2l_B+D-1)\left[\frac{\beta_{11}}{4\beta_{12}}(4l_B+2D-3)\right]^3$	$\Psi_B(r)=N_B r^{l_B}\exp\left[-\sqrt{-E_b}\,r-\frac{\beta_{11}\sqrt{r}}{\sqrt{-E_B}}\right]$
$\alpha_2 g_C^{-\frac{2}{3}}$	$V_C=\beta_{21}r^{-1}+\beta_{22}r+\beta_{23}r^2$; $\beta_{21}=\frac{3\alpha_3}{2}\left(\frac{-E_C}{\alpha_2}\right)^{\frac{1}{2}}$, $\beta_{22}=\frac{2(-E_A)}{3}\left[\frac{-E_C}{\alpha_2}\right]^{\frac{3}{2}}$, $\beta_{23}=\alpha_1\left[\frac{-2E_C}{3\alpha_2}\right]^2$.	$E_C=-\left[\frac{\beta_{22}^{\ 2}}{4\beta_{23}}-(2l_C+D)\sqrt{\beta_{23}}\right]$	$\beta_{21}=-\left(l_C+\frac{D-1}{2}\right)\frac{\beta_{22}}{\sqrt{\beta_{23}}}$	$\Psi_C(r)=N_C r^{l_C}\exp\left[-\frac{\sqrt{\beta_{23}}}{2}r^2-\frac{\beta_{22}}{2\sqrt{\beta_{23}}}r\right]$
$\alpha_3 g_D^{-\frac{4}{3}}$	$V_D(r)=\beta_{31}r^6+\beta_{32}r^4+\beta_{33}r^2$ $\beta_{31}=9\alpha_1\left(\frac{E_D}{9\alpha_3}\right)^4,\beta_{32}=\frac{E_A}{81}\left(\frac{E_D}{\alpha_3}\right)^3$, $\beta_{33}=\alpha_2\left(\frac{E_D}{3\alpha_3}\right)^2$	$E_D=\frac{\beta_{32}}{\sqrt{\beta_{31}}}\left(l_D+\frac{D}{2}\right)$	$\beta_{32}=2\sqrt{\beta_{31}}$ $\left[(2l_D+D+2)\sqrt{\beta_{31}}+\beta_{33}\right]$	$\Psi_D(r)=N_D r^{l_D}\exp\left[-\frac{\sqrt{\beta_{31}}}{4}r^4-\frac{\beta_{32}}{4\sqrt{\beta_{31}}}r^2\right]$

Integral power Singular Multiterm self-dual potential

We have considered the following exactly solved quantum system [12] to investigate the self-duality of a potential and the related condition for self-duality:

$$V_A(r)=\alpha_1 r^2+\alpha_2 r^{-4}+\alpha_3 r^{-6}, \qquad \alpha_1>0; \quad \alpha_3>0. \tag{53}$$

The EAS is provided by [14]

$$\Psi_A(r)=N_A r^{\frac{3+\frac{\alpha_2}{\sqrt{\alpha_3}}}{2}}\exp\left[-\frac{1}{2}\left(\sqrt{\alpha_1}r^2+\sqrt{\alpha_3}r^{-2}\right)\right] \tag{54}$$

With a constraint

$$\left(2\sqrt{\alpha_3}+\alpha_2\right)^2=\alpha_3\left[(2l_A+1)^2+8\sqrt{\alpha_3\alpha_1}\right]. \tag{55}$$

The energy eigenvalue for the potential system is provided by [14] and is:

$$E_A=\sqrt{\alpha_1}\left(4+\frac{\alpha_2}{\sqrt{\alpha_3}}\right). \tag{56}$$

Selecting the working potential $V_A^W\left(g_B(r)\right)=\alpha_2 g_B{}^{-4}$ and utilizing Eq. (16), which yields

$$g_B(r)=-\frac{1}{\left(-\frac{E_B}{\alpha_2}\right)^{\frac{1}{2}}r}. \tag{57}$$

Applying Eqs. (12) and (13) we have the following potential of B-QS:

$$V_B(r)=\beta_{11}r^2+\beta_{12}r^{-4}+\beta_{13}r^{-6} \tag{58}$$

Where the parameters of the potential are:

$$\beta_{12}=C_B{}^2=\frac{\alpha_2 E_A}{E_B} \tag{59}$$

$$\beta_{11}=\frac{\alpha_3 E_B{}^2}{\alpha_2{}^2}, \quad \beta_{13}=\frac{\alpha_1\alpha_2{}^2}{E_B{}^2}. \tag{60}$$

The relation between the angular momentum quantum numbers l_A and l_B of A-QS and B-QS

is obtained from Eq. (14). Energy eigenvalue of B-QS from eq. (59) comes out to be:

$$E_B=\sqrt{\beta_{11}}\left(4+\frac{\beta_{12}}{\sqrt{\beta_{13}}}\right) \tag{61}$$

With the constraint equation

$$\left(2+\frac{\beta_{12}}{\sqrt{\beta_{13}}}\right)^2=(2l_B+D-2)^2+8\sqrt{\beta_{11}}\sqrt{\beta_{13}}. \tag{62}$$

The exact energy eigenfunction of B-QS is obtained from eq. (3) as:

$$\Psi_B(r)=N_B r^{\frac{4-D+\frac{\beta_{12}}{\sqrt{\beta_{13}}}}{2}} \exp\left[-\frac{1}{2}\left(\sqrt{\beta_{11}}r^2+\sqrt{\beta_{13}}r^{-2}\right)\right]. \tag{63}$$

Referring to eq. (52) one finds that if the transformed B-QS is identical to the untransformed A-QS for a particular working potential say, $\alpha_1 r^{a_1}$, implies that $A_1 = B_1$ which fixes $a_1 = 0,-4$. For non- trivial transformation, $a_1 = -4.$ A further consequence is:

$$a_2 + a_3 = -4. \tag{64}$$

A three term potential, therefore will be self dual under ET when the working potential chosen has the exponent (-4), and the sum of the other two exponents is again (-4). This immediately rules out any self-dual two terms potential.

The behaviour of the potential and some of the generated EAS of **C-QS** are given in Fig.3.

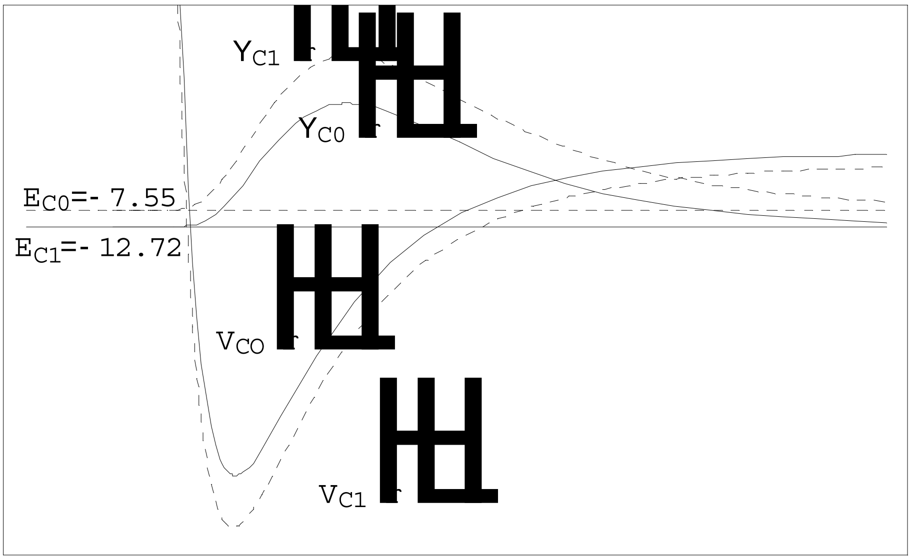

Figure 3: Continuous curves are for the EQS $\{E_{CO},V_{CO}(r),\Psi_{C0}(r)\}$, where the parameter set is (E_{C0}=-12.72, β_{21}=-17.8, β_{22}=1.5, β_{23}=-8.57, l_C=0, D=3) and the broken curves are for the EQS $\{E_{C1}, V_{C1}(r), \Psi_{C1}(r)\}$, where the parameter set is (E_{C1}=-7.55, β_{21}=-13.728, β_{22}=1.5, β_{23}=-8.57, l_C=1, D=3). The graphs are drawn in arbitrary scale.

Table 3: List of generated exactly solved potentials (ESPs) in D-dimensional Euclidean spaces from the known (A-QS) integral power singular potential: $V_A(r) = \alpha_1 r^2 + \alpha_2 r^{-4} + \alpha_3 r^{-6}$.

WP	Exactly solved potential	Energy Eigenvalue	Constraint Equation	Energy Eigen Function
$\alpha_2 g_B^{-4}$	$V_B(r) = \beta_{11} r^2 + \beta_{12} r^{-4} + \beta_{13} r^{-6}$ *Self-dual potential*	$E_B = -\sqrt{\beta_{11}}\left(4 + \frac{\beta_{12}}{\sqrt{\beta_{13}}}\right)$	$\left[2 + \frac{\beta_{12}}{\sqrt{\beta_{13}}}\right]^2 = (2l_B + D - 2)^2 + 8\sqrt{\beta_{13}\beta_{11}}$	$\Psi_B(r) = N_B r^{\frac{4 - D + \frac{\beta_{12}}{\sqrt{\beta_{13}}}}{2}} \exp\left[-\frac{1}{2}\left(\sqrt{\beta_{11}} r^2 + \sqrt{\beta_{13}} r^{-2}\right)\right]$
$\alpha_1 g_C^{2}$	$V_C(r) = -\beta_{21} r^{-1} + \beta_{22} r^{-4} + \beta_{23} r^{-3};$ $\beta_{21} = \frac{E_A}{2}\left[-\frac{E_C}{\alpha_1}\right]^{\frac{1}{2}}, \beta_{22} = \frac{\alpha_3}{16}\left[-\frac{E_C}{\alpha_1}\right]^{-1}$ $\beta_{23} = \frac{\alpha_2}{8}\left[-\frac{E_C}{\alpha_1}\right]^{-\frac{1}{2}}$	$E_C = -\frac{\beta_{21}^2}{\left(2 + \frac{\beta_{23}}{\sqrt{\beta_{22}}}\right)^2}$	$\left[1 + \frac{\beta_{23}}{\sqrt{\beta_{22}}}\right] = (2l_C + D - 2)^2 + \frac{8\beta_{21}\sqrt{\beta_{22}}}{2 + \frac{\beta_{23}}{\sqrt{\beta_{22}}}}$	$\Psi_C(r) = N_C r^{\frac{3 - D + \frac{\beta_{23}}{\sqrt{\beta_{22}}}}{2}} \exp\left[-\frac{\beta_{21}}{2 + \frac{\beta_{23}}{\sqrt{\beta_{22}}}} r - \frac{\sqrt{\beta_{22}}}{r}\right]$
$\alpha_3 g_D^{-6}$	$V_D = \beta_{31} r^{-1} + \beta_{32} r^{-4} + \beta_{33} r^{-3}$ $\beta_{31} = \frac{\alpha_2}{2}\left[-\frac{\alpha_3}{E_D}\right]^{-\frac{1}{2}}, \beta_{32} = -\frac{\alpha_3 \alpha_1}{16 E_D}$ $\beta_{33} = \frac{-E_A}{8}\left[-\frac{\alpha_3}{E_D}\right]^{\frac{1}{2}}$	$E_D = -\left[\frac{\beta_{31}\sqrt{\beta_{32}}}{\beta_{33} + 2\sqrt{\beta_{32}}}\right]^2$	$\left[1 + \frac{\beta_{31}}{\sqrt{-E_D}}\right]^2 = (2l_D + D - 2)^2 + 8\sqrt{-E_D}\sqrt{\beta_{32}}$	$\Psi_D(r) = N_D r^{-\left(\frac{\beta_{31} + (D-1)\sqrt{-E_D}}{2\sqrt{-E_D}}\right)} \exp\left[-\frac{\sqrt{\beta_{32}}}{r} + \sqrt{-E_D}\, r\right]$

Conclusions

For quantum multiterm potentials it is possible to generate a finite number of different exactly solved quantum systems by selecting the working potential, as mentioned earlier. We however restrict ourselves to taking one term working potential. Two or multiterm working potentials are not considered as they offer the following practical difficulties: (i) the integral specifying the transformation function $g_B(r)$ can not be extracted analytically in most of the cases, (ii) even if such integrals are found they are of the form $F(r) = r + c$ and the analytical inverse function $F^{-1}(g)$ can not be found. Here we have reported two different and three different exactly solved potentials generated formed exactly solved (a) two terms fractional power potentials, (b) three term fractional power potentials and (c) integral power singular potential respectively. The daughter along with the parent potential formed family relationship (multiplet structure) as one can go form one ESP to the other ESPs with the help of ET as shown in Fig. 4- 6. The multiplet structure however is no related to representation of any group, as ET doesn't form a group in the conventional sense. In Fig.4 the arrows indicate that the working potential $\alpha_2 r^{-\frac{3}{2}}$ and energy eigenvalue lead to the energy eigenvalue and the potential term of B-QS respectively. The remaining term(s) of the potential constitute the different term(s) in the generated potential. The double headed arrow indicates that if $\beta_{11} r^6$ is chosen as working potential of the B-QS we could revert beck to A-QS exactly. Likewise for other potential terms of Fig.4. The same is true for Figs.5, 6 respectively. Further unlike a single term ESP, for two term and /multiterm ESP the solvability requires that the constant coefficients of the different terms satisfy a constraint equation. It is noteworthy that under ET

The constraint equation gets converted in to the energy eigenvalue expression and the energy eigenvalue in to a constraint equation. For self-dual potential it is found that one of the exponents of the three terms potential must be (-4) and sum of the other two exponents must again be (-4), which automatically rules out two- term self dual potentials. In view of relations (52) given for three term potentials or generalization of Eq. (52) for multiterm potentials it is easy to check that two term self-dual potentials must be of the form $\alpha_1 r^{-4} + \alpha_2 r^{-2}$, where the second term absorbed centrifugal barrier term which makes its essentially $\alpha_1 r^{-4}$, which is a monoterm self-dual potential. Self-duality constraints that only monoterm and three term self dual potential are possible.

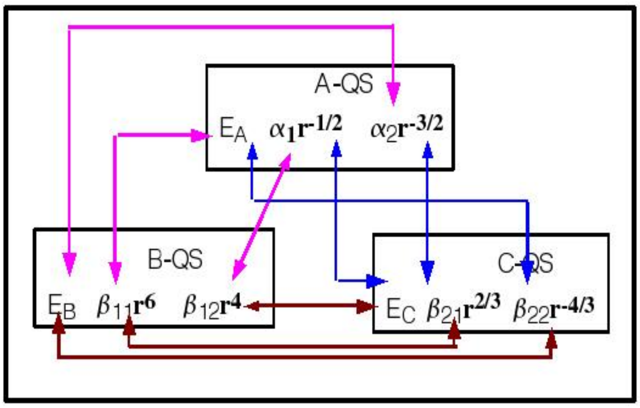

Figure 4: The family relationship of the generated EQS for the two term fractional power potential.

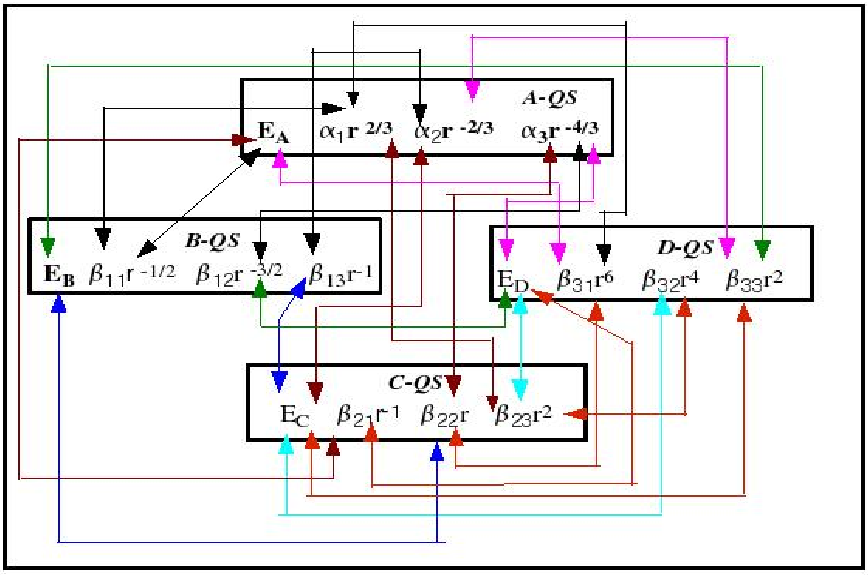

Figure 5: The family relationship of the generated EQS for the three term fractional power potential. (Explanation of Fig. is given in the discussion/conclusion.)

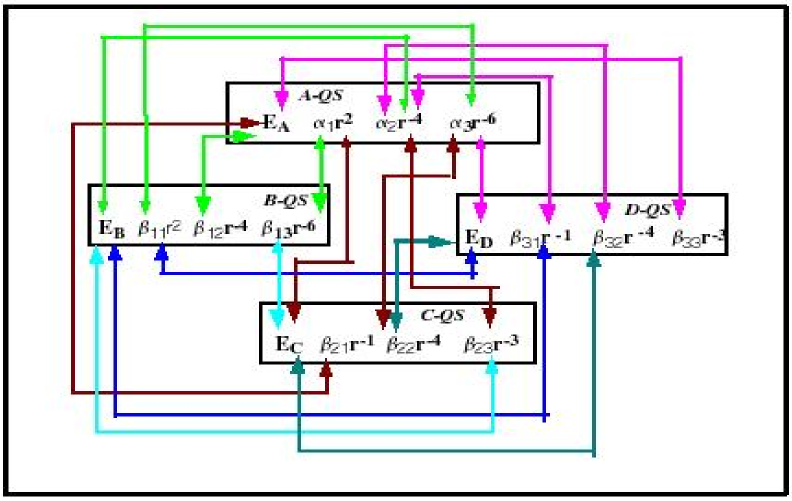

Figure 6: The family relationship of the generated EQS for the Integral power singular potential.

References

[1] Flessas, G.P., 1979, "Exact Solutions for a doubly anharmonic oscillator," Phys.Lett, **A72**, pp289.

[2] Singh.,V., Biswas, S.N., and Datta,K., 1978, “Anharmonic Oscillator and analytic theory of Continued fractions,” Phys.Rev., **D18**, pp.1901

[3] Flessas, G.P., and Watt, A., 1981, “An exact solution of the Schrödinger equation for a multiterm potential,” J.Phys. A: Math.Gen., **14**, pp L315-L318

[4] Flessas, G.P., and Das, K.P.,1981, ” Exact solution for anharmonic Oscillators,”J.Phys.A:Math.Gen., **14**, pp L209-L211

[5] Roy, P., and Roychoudhury, R., 1987, “On exact solutions of the doubly anharmonic oscillator,” J.Phys.A:Math.Gen.**20** pp.6597-6601. “

[6] Adhikari, R., Dutt, R., and Varshni, Y. P., 1989, “Exact Solution for the Polynomial Potentials using Supersymmetry Factorization Method,” Phys.Lett., **A 141**, pp 1-8.

[7] Ahmed, S.A.S., 1997, “A Transformation Method of Generating exact analytic Solutions of the Schrödinger Equation,” International Journal of Theoretical physics, **36**, pp. 1893.

[8] Ahmed, S.A.S. Ahmed, Borah, B.C., and Sarma, D., 2001,” Generation of exact bound state solutions from solvable non-power law potentials by a transformation method,” Eur.Phys J., **D17**, pp. 5-11.

[9] Bose, S.K., 1994, “Exact Bound States for the central fraction power singular Potential,” Nuovo Cimento, **109B**, pp. 1217-1220.

[10] Bose, S.K., 1996, “Exact solution of non-relativistic Schrödinger Equation for Certain Central physical potentials,”, Proceedings of Int. colloquium on Group Theoretical Methods in Physics at Goslar, Germany.

[11] Dong, Shi-Hai., and Ma, Zhong-Qi, 1998, “Exact Solutions to the Schrödinger equation on the Potential,” J.Phys. A: Math.Gen. **31**, pp.9855-9859.

[12] Znojil, M., 1982,”Elementary bound states for the Power law Potentials,”, J.Phys.A: Math.Gen,**15**,pp.2111-2122

[13] Dutra,A.de Souza.,1993, “Conditionally exactly soluble class of quantum potentials,” 1993, Phys.Rev., **A 47**, pp 2435-2437

[14] Raushal, R.S., and Parashar, D., 1992, “On the quantum bound states for the potential $V(r) = ar^2 + br^{-4} + cr^{-6}$,” Phys.Lett, **A170**, pp 335-338.

[15] Hille, E., 1969, Lecturers on ordinary differential equations (Reading MA.: Addison Wesley)p.647

[16] Gradshteyn I.S. and Ryzhik I.M., Table of integrals, series and products (Academic Press, New York, 1965) p.342.