

Some New Properties of Generalized Voigt - Function

Atul Garg[#] and Yashwant Singh^{*}

[#] *Department of mathematics, S.M.L. (PG) College,
Jhunjhunu 333001, Rajasthan, India
E-mail: atuljgn@yahoo.co.in*

^{*} *Department of mathematics, Jubail University College,
Jubail industrial city, Kingdom of Saudi Arabia
E-mail: ysingh23@yahoo.co.in*

Abstract

This paper deals with a special class of functions called generalized voigt function $V_{\mu,\nu}(x, y)$. It plays an important role in the several diverse fields of physics such as plasma spectroscopy, astrophysics, nuclear magnetic resonance and the theory of neutron reactions etc.

Some new properties of generalized voigt-function are introduced here to facilitate the computation of $V_{\mu,\nu}(x, y)$ when parameters or arguments take different values.

AMS Subject Classification: 33E20, 85A99

Key Words: Voigt function, Bessel function

Introduction

The Voigt function $K(x,y)$ and $L(x,y)$ occur frequently in a wide variety of problems of physics such as astrophysical spectroscopy, transfer of radiation on heated atmosphere and also in the theory of neutron reactions [1]. Further more, the function

$$K(x,y) + i L(x,y)$$

is, to a numerical factor, identical to the so-called ‘plasma dispersion function’, which is tabulated by Fried and Conte [6] and several representations of the Voigt functions have been given by a number of workers, for examples, Exton [4], Reiche [9], Fetti [5], Shrivastava and Miller [11], and Srivastava et. al. [12] etc. First of all, we recall here the following integral representations, due to Reiche [9]:

$$K(x,y) = (\pi)^{\frac{-1}{2}} \int_0^{\infty} \exp\left(-yt - \frac{1}{4}t^2\right) \cos(xt) dt, \quad (1.1)$$

and

$$L(x,y) = (\pi)^{\frac{-1}{2}} \int_0^{\infty} \exp\left(-yt - \frac{1}{4}t^2\right) \sin(xt) dt \quad (-\infty < x < \infty, \quad y > 0), \quad (1.2)$$

so that

$$\begin{aligned} K(x,y) \pm i L(x,y) &= (\pi)^{\frac{-1}{2}} \int_0^{\infty} \exp\left(-\left(y \mp ix\right)t - \frac{1}{4}t^2\right) dt \\ &= \exp\left[\left(y \mp ix\right)^2\right] \{1 - \operatorname{erf}\left(y \mp ix\right)\} \end{aligned} \quad (1.3)$$

Where $\operatorname{erf}(w)$ denotes the error function [10,p.28].

Srivastava and Miller [11,(8),p. 113] introduced and studied systematically a unification (and generalization) of the Voigt function $K(x,y)$ and $L(x,y)$ in the form

$$V_{\mu,\nu}(x,y) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \int_0^{\infty} t^{\mu} \exp\left(-yt - \frac{1}{4}t^2\right) J_{\nu}(xt) dt \quad (1.4)$$

so that

$$V_{1/2,-1/2}(x,y) = K(x,y) \quad \text{and} \quad V_{1/2,1/2}(x,y) = L(x,y), \quad (1.5)$$

where the Bessel function $J_{\nu}(x)$ of order ν is defined by

$$J_{\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{\nu+2m}}{m! \Gamma(\nu+m+1)}, \quad |x| < \infty \quad (1.6)$$

Also [13]

$$2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \quad (1.7)$$

$$J_n(-x) = (-1)^n J_n(x) \quad (1.8)$$

$$\sqrt{\frac{\pi x}{2}} J_{\frac{1}{2}}(x) = \operatorname{Sin} x \quad (1.9)$$

$$\sqrt{\frac{\pi x}{2}} J_{\frac{3}{2}}(x) = \frac{\operatorname{Sin} x}{x} - \operatorname{Cos} x \quad (1.10)$$

Result I

$$2\nu V_{\mu,\nu}(x,y) = x[V_{\mu+1,\nu+1}(x,y) + V_{\mu+1,\nu-1}(x,y)] \quad (2.1)$$

Proof

Replacing x by xt and n by ν in (1.7), we get

$$2\nu J_{\nu}(x) = xt [J_{\nu-1}(xt) + J_{\nu+1}(xt)]$$

On multiplying both side by $t^\mu \exp(-yt - \frac{1}{4}t^2)$ and integrating between 0 to ∞ with respect to t, we get

$$2\nu \int_0^\infty t^\mu \exp(-yt - \frac{1}{4}t^2) J_\nu(xt) dt = \int_0^\infty xt.t^\mu \exp(-yt - \frac{1}{4}t^2) [J_{\nu-1}(xt) + J_{\nu+1}(xt)] dt$$

Now by virtue of definition of generalized Voigt function (1.4), result (2.1) follows immediately.

Result

$$V_{\mu,\nu}(-x, y) = (-1)^n i V_{\mu,\nu}(x, y) \tag{2.2}$$

Proof

Changing the sign of x in (1.4) i.e. in the definition

$$V_{\mu,\nu}(-x, y) = (\frac{-x}{2})^{\frac{1}{2}} \int_0^\infty t^\mu \exp(-yt - \frac{1}{4}t^2) J_\nu(-xt) dt$$

with the help of (1.8), we get

$$V_{\mu,\nu}(-x, y) = i (\frac{x}{2})^{\frac{1}{2}} \int_0^\infty t^\mu \exp(-yt - \frac{1}{4}t^2) (-1)^n J_\nu(xt) dt$$

Now by virtue of definition of generalized Voigt function $V_{\mu,\nu}(x, y)$, required result (2.2) obtained easily.

Result

$$V_{\mu,\nu}(x, y) = (-1)^{n+1} i V_{\mu,\nu}(-x, y) \tag{2.3}$$

Proof

On simply dividing result (2.2) by $(-1)^n i$, result (2.3) obtained easily

Result

$$V_{-\frac{1}{2}, \frac{1}{2}}(x, y) = x \left[V_{\frac{1}{2}, \frac{3}{2}}(x, y) + K(x, y) \right] \tag{2.4}$$

Proof

After replacing x by xt, multiplying both the side of (1.10) by $e^{(-yt - \frac{1}{4}t^2)}$ and integrating with respect to t between 0 to ∞ , we get

$$\int_0^\infty e^{(-yt - \frac{1}{4}t^2)} \sqrt{\frac{\pi xt}{2}} J_{\frac{3}{2}}(xt) dt = \int_0^\infty e^{(-yt - \frac{1}{4}t^2)} \frac{\text{Sin } xt}{xt} dt - \int_0^\infty e^{(-yt - \frac{1}{4}t^2)} \text{Cos } xt dt$$

On using (1.1),

$$(\frac{x}{2})^{\frac{1}{2}} \sqrt{\pi} \int_0^\infty t^{\frac{1}{2}} .e^{(-yt - \frac{1}{4}t^2)} J_{\frac{3}{2}}(xt) dt = \int_0^\infty e^{(-yt - \frac{1}{4}t^2)} \frac{\text{Sin } xt}{xt} dt - \sqrt{\pi} K(x, y)$$

Now using (1.9), we get

$$(\frac{x}{2})^{\frac{1}{2}} \int_0^\infty t^{\frac{1}{2}} .e^{(-yt - \frac{1}{4}t^2)} J_{\frac{3}{2}}(xt) dt = \frac{1}{\sqrt{2x}} \int_0^\infty t^{\frac{1}{2}} e^{(-yt - \frac{1}{4}t^2)} J_{\frac{1}{2}}(xt) dt - K(x, y)$$

By virtue of definition of generalized Voigt function (1.4), result (2.4) follows immediately.

Result

$$V_{-\frac{1}{2}, -\frac{1}{2}}(x, y) = x \left[L(x, y) + V_{\frac{1}{2}, -\frac{1}{2}}(x, y) \right] \quad (2.5)$$

Proof

Result (2.4) can also be obtained directly by putting $\mu = \frac{-1}{2}$ and $\nu = \frac{1}{2}$ in recurrence relation (2.1). So by putting $\mu = \frac{-1}{2}$ and $\nu = \frac{-1}{2}$ in (2.1), we come to the last value of generalized voigt function {see (1.5)} when both the parameters μ, ν have value $\frac{1}{2}$ with \pm sign.

Acknowledgment

The present investigation is supported by the UGC under grant MS – 158 / 304044 / 07-08 / CRO.

References

- [1] B.M. Armstrong, R.W. Nicholls ; Emission, Absorption and Transfer of Radiation in Heated Atmospheres, Pergamon Press, New York, 1972.
- [2] Yu.A. Brychkov, H.J. Glaeske, A.P. Prudnikov, Vu.K. Tuan; Multidimensional Integral Transformations, Gordon & Breach, Philadelphia, 1992.
- [3] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi; Tables of Integral Transforms, vol.I, McGraw-Hill, New York, 1954.
- [4] H. Exton; On reducibility of the Voigt function ; J. Phis. A: Math. Gen. 14(4)(1981), 175-177.
- [5] H.E. Fettis; Remark on a note by Exton, H., On reducibility of the Voigt function, J. Phis. A: Math. Gen. 16(1983), 663-664.
- [6] B.D. Fried, S.D. Conte; The Plasma Dispersion Function, Academic Press, New York, 1961.
- [7] M. Kumarujama, D. Singh; Some representations of unified Voigt functions (communicated).
- [8] E.D. Rainville; Special Functions, The McMillan Company, New York, 1960.
- [9] F. Reiche; Uber die emission, absorption and intensitatsverteilung von spektrallinien, Ber. Deutsgh Phys. Gen., 15(1913), 3-21.
- [10] H.M. Srivastava, H.L. Manocha; A Tretise on Generating Functions, Halsted Press (Ellis Horwood Ltd., Chichester), New York, 1984.
- [11] H.M. Srivastava, E.A. Miller; A unified presentation of the Voigt functions; Astrophys. Space Sci. 135(1987), 111-115.
- [12] H.M. Srivastava, M.A. Pathan and M. Kumarujama; Some unified presentations of the generalized Voigt functions, Comm. Appl. Anal., 2(1)(1998), 49-64.
- [13] M.A. Pathan, P.K. Benargi; Special Functions and Calculus of Variations, RBD, New Delhi, 2004.