

Knots in Condensed Matters

Y. M. Cho

*Center for Theoretical Physics and School of Physics
College of Natural Sciences, Seoul National University, Seoul 151-742, Korea
E-mail: ymcho@yongmin.snu.ac.kr*

Abstract

We propose two types of topologically stable knot solitons in condensed matters, one in two-component Bose-Einstein condensates and one in two-gap superconductors. We identify the knot in Bose-Einstein condensates as a twisted vorticity flux ring and the knot in two-gap superconductors as a twisted magnetic flux ring. In both cases we show that there is a remarkable interplay between topology and dynamics which transforms the topological stability to the dynamical stability, and vice versa. We discuss how these knots can be constructed in the spin-1/2 condensate of ^{87}Rb atoms and in two-gap superconductor of MgB_2 .

Keywords: Topological knot in BEC, Knot in two-gap superconductor.

1. Introduction

Topological objects (monopoles, vortices, skyrmions, etc.) have played an increasingly important role in physics [1, 2]. In particular finite energy topological objects have been widely studied in theoretical physics [3, 4]. A recent addition to this family of finite energy solitons has been the knots [5, 6]. The interest on these topological objects, however, has been confined mainly to theoretical physics, because most of them (except the vortices) exist in “hypothetical” worlds which are very difficult to create in laboratories. The only topological objects which one can realistically expect to exist in the “standard” models are the electroweak monopoles and dyons in Weinberg-Salam model which have a non-trivial W_μ and Z_μ dressing [7, 8]. Unfortunately these objects could carry an infinite energy, which makes it impossible to create them experimentally.

Fortunately the recent experimental advances in condensed matter physics, in particular the construction of multi-component Bose-Einstein condensates (BEC) made of ^{87}Rb [9, 10] and two-gap superconductors made of MgB_2 [11, 12], have widely opened new opportunities for us to create new topological objects experimentally which so far have been only of theoretical interest. The purpose of this paper is to argue that these

new multi-component condensates could allow us to have real knots, topologically stable finite energy 3-dimensional solitons. This is because, due to the multi-component structure, the new condensates have a non-Abelian symmetry which provides the needed topology for the stable knots.

To understand how the realistic knots can appear in these condensed matters, it is necessary to understand the prototype Faddeev-Niemi knot in Skyrme theory. The Skyrme theory is well known to have a magnetic vortex known as the baby skyrmion [13], and the Faddeev-Niemi knot can be identified as a twisted vortex ring made of a helical baby skyrmion [6, 14]. In the following we show how one can construct similar knots from helical vortices in two-component Bose-Einstein condensates and two-gap superconductors.

The paper is organized as follows. In Section II we discuss how the helical vortex can give rise to the prototype knot in Skyrme theory for later purpose. In Section III we discuss gauge theory of two-component BEC which has the vorticity interaction, and show that the theory is closely related to Skyrme theory. With this we show that the theory allows a topological knot similar to the one in Skyrme theory. We also identify that this knot is a vorticity knot very similar to the one in Gross-Pitaevskii theory of two-component BEC. In Section IV we discuss the Landau-Ginzburg theory of two-gap superconductor and show that the theory is closely related to the gauge theory of two-component BEC. With this we argue that the theory can also admit a knot, a twisted magnetic vortex ring, similar to the vorticity knot in two-component BEC. In doing so we also establish the non-Abelian flux quantization in two-gap superconductor. We demonstrate the existence of magnetic vortex whose flux is quantized in the unit $4\pi/g$, not $2\pi/g$, in two-gap superconductor. Finally in Section V we discuss physical implications of our results.

2. Knot in Skyrme theory

The Skyrme theory has a rich topological structure. It has skyrmion and baby skyrmion [3, 13]. But recently it has been shown that the theory allows a knotted soliton identical to the Faddeev-Niemi knot in Skyrme-Faddeev non-linear sigma model, which can be identified as a twisted vortex ring made of helical baby skyrmion [6, 14]. To see this, we review the knot in Skyrme theory first.

Let ω and \hat{n} be the scalar field and the non-linear sigma field in Skyrme theory. With

$$U = \exp(i\vec{\sigma} \cdot \hat{n} \frac{\omega}{2}) = \cos \frac{\omega}{2} - i(\vec{\sigma} \cdot \hat{n}) \sin \frac{\omega}{2}, \quad L_\mu = U \partial_\mu U^\dagger, \quad (\hat{n}^2 = 1) \quad (1)$$

the Skyrme Lagrangian is expressed as

$$\begin{aligned} \mathcal{L} &= \frac{\mu^2}{4} \text{tr} L_\mu^2 + \frac{\alpha}{32} \text{tr} ([L_\mu, L_\nu])^2 \\ &= -\frac{\mu^2}{4} \left[\frac{1}{2} (\partial_\mu \omega)^2 + (1 - \cos \omega) (\partial_\mu \hat{n})^2 \right] \end{aligned}$$

$$-\frac{\alpha}{16} \left[\frac{1 - \cos \omega}{2} (\partial_\mu \omega \partial_\nu \hat{n} - \partial_\nu \omega \partial_\mu \hat{n})^2 + (1 - \cos \omega)^2 (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2 \right]. \quad (2)$$

The equation of motion is given by

$$\begin{aligned} & \frac{\mu^2}{4} [\partial^2 \omega - \sin \omega (\partial_\mu \hat{n})^2] + \frac{\alpha}{32} \sin \omega (\partial_\mu \omega \partial_\nu \hat{n} - \partial_\nu \omega \partial_\mu \hat{n})^2 \\ & + \frac{\alpha}{8} (1 - \cos \omega) \partial_\mu [(\partial_\mu \omega \partial_\nu \hat{n} - \partial_\nu \omega \partial_\mu \hat{n}) \cdot \partial_\nu \hat{n}] - \frac{\alpha}{8} (1 - \cos \omega) \sin \omega (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2 = 0, \\ & \partial_\mu \left\{ \frac{\mu^2}{4} (1 - \cos \omega) \hat{n} \times \partial_\mu \hat{n} + \frac{\alpha}{16} (1 - \cos \omega) [(\partial_\nu \omega)^2 \hat{n} \times \partial_\mu \hat{n} - (\partial_\mu \omega \partial_\nu \omega) \hat{n} \times \partial_\nu \hat{n}] \right. \\ & \left. + \frac{\alpha}{8} (1 - \cos \omega)^2 (\hat{n} \cdot \partial_\mu \hat{n} \times \partial_\nu \hat{n}) \partial_\nu \hat{n} \right\} = 0. \end{aligned} \quad (3)$$

With the spherically symmetric ansatz

$$\omega = \omega(r), \quad \hat{n} = \hat{r}, \quad (4)$$

(3) is reduced to

$$\begin{aligned} & \frac{d^2 \omega}{dr^2} + \frac{2}{r} \frac{d\omega}{dr} - \frac{2 \sin \omega}{r^2} - \frac{\alpha}{\mu^2} \left[\frac{\sin^2(\omega/2)}{r^2} \frac{d^2 \omega}{dr^2} + \frac{\sin \omega}{2r^2} \left(\frac{d\omega}{dr} \right)^2 - \frac{2 \sin \omega \sin^2(\omega/2)}{r^4} \right] \\ & = 0. \end{aligned} \quad (5)$$

Imposing the boundary condition $\omega(0) = 2\pi$ and $\omega(\infty) = 0$, one has the well-known skyrmion [3].

A remarkable point of Skyrme theory is that $\omega = \pi$, independent of \hat{n} , becomes a classical solution [6]. So restricting ω to π , one can reduce the Lagrangian (2) to the Skyrme-Faddeev Lagrangian

$$\mathcal{L} \rightarrow -\frac{\mu^2}{2} (\partial_\mu \hat{n})^2 - \frac{\alpha}{4} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2, \quad (6)$$

in which case the equation of motion (3) is reduced to

$$\begin{aligned} & \hat{n} \times \partial^2 \hat{n} + \frac{\alpha}{\mu^2} (\partial_\mu N_{\mu\nu}) \partial_\nu \hat{n} = 0, \\ & N_{\mu\nu} = \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu. \end{aligned} \quad (7)$$

Notice that since $N_{\mu\nu}$ forms a closed two-form, it always admits a $U(1)$ potential C_μ .

The equation (7) allows non-Abelian monopole, baby skyrmion, helical baby skyrmion, and Faddeev-Niemi knot as its solutions [6, 14]. But for our purpose it is important to understand the helical baby skyrmion, because this plays a crucial role for us to construct the knot. So we review the helical baby skyrmion.

To construct the desired helical baby skyrmion let (ρ, φ, z) the cylindrical coordinates, and choose the ansatz

$$\hat{n} = \begin{pmatrix} \sin f(\rho) \cos (mkz + n\varphi) \\ \sin f(\rho) \sin (mkz + n\varphi) \\ \cos f(\rho) \end{pmatrix}. \quad (8)$$

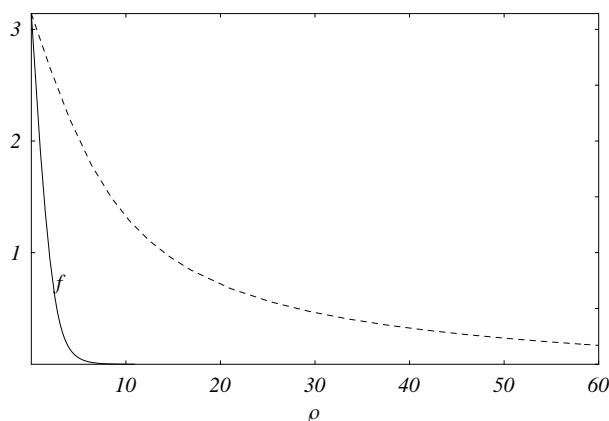


Figure 1: The baby skyrmion (dashed line) with $m = 0, n = 1$ and the helical baby skyrmion (solid line) with $m = n = 1$ in Skyrme theory. Here ρ is in the unit $\sqrt{\alpha}/\mu$ and $k = 0.8 \mu/\sqrt{\alpha}$.

With this the equation (7) is reduced to

$$\begin{aligned} & \left(1 + (m^2 k^2 + \frac{n^2}{\rho^2}) \frac{\sin^2 f}{g^2 \rho^2}\right) \ddot{f} + \left(\frac{1}{\rho} + 2\frac{\dot{\rho}}{\rho} + (m^2 k^2 + \frac{n^2}{\rho^2}) \frac{\sin f \cos f}{g^2 \rho^2}\right) \dot{f} \\ & + \frac{1}{\rho} (m^2 k^2 - \frac{n^2}{\rho^2}) \frac{\sin^2 f}{g^2 \rho^2} f - (m^2 k^2 + \frac{n^2}{\rho^2}) \sin f \cos f = 0. \end{aligned} \quad (9)$$

So with the boundary condition

$$f(0) = \pi, \quad f(\infty) = 0, \quad (10)$$

we obtain the non-Abelian vortex solutions shown in Fig.1. Notice that, when $m = 0$, the solution describes the well-known baby skyrmion [13]. But when m is not zero, it describes a helical baby skyrmion, a twisted magnetic vortex which is periodic in z -coordinate [14]. In this case, the vortex has a quantized magnetic flux not only along the z -axis but also around the z -axis.

The helical baby skyrmion will become unstable unless the periodicity condition is enforced by hand. But it plays a very important role because one can make it a vortex ring by smoothly connecting two periodic ends. In this case the vortex ring acquires the topology of a knot, and thus becomes a knot itself [6, 14]. In fact it becomes a knot made of two magnetic fluxes linked together, whose knot topology is described by the Chern-Simon index of the potential C_μ ,

$$Q = \frac{1}{32\pi^2} \int \varepsilon_{ijk} C_i N_{jk} d^3x = mn, \quad (11)$$

which describes the Hopf mapping $\pi_3(S^2)$ defined by \hat{n} .

Clearly the knot has a topological stability, because two flux rings linked together can not be disjointed by a smooth deformation of the field configuration. Moreover the topological stability is backed up by a dynamical stability. This is because the knot can be viewed as two magnetic fluxes linked together, and the magnetic flux trapped in the knot disk can not be squeezed out. This provides a stabilizing repulsive force which prevent the collapse of the knot [14].

From our discussion it becomes clear that the existence of the helical baby skyrmion is crucial for the existence of a topologically stable knot. In the following we show that an identical mechanism works in two-component BEC and two-gap superconductor which guarantees the existence of a stable knot.

3. Knot in Two-component Bose-Einstein Condensates

The recent advent of multi-component BEC (in particular the spin-1/2 condensate of ^{87}Rb atoms) has widely opened a new opportunity for us to study novel topological objects which can not be realized in ordinary (one-component) BEC [9, 10]. This is because the multi-component BEC naturally allows a non-Abelian structure which accommodates a non-trivial topological objects, in particular a topological knot which is very similar to the knot in Skyrme theory [15, 16].

There are two competing theories of two-component BEC, the popular Gross-Pitaevskii theory [10] and the gauge theory of two-component BEC proposed recently [15]. Both theories predict topological knots. Many authors have already claimed the existence of a knot in Gross-Pitaevskii theory [10]. Moreover it has been shown that this knot can be identified as a vorticity knot which is made of two vorticity fluxes linked together, whose topology $\pi_3(S^2)$ is fixed by the Chern-Simon index of the velocity potential of the condensate [16].

So in this section we discuss the gauge theory of two-component BEC, and show that this theory also has a vorticity knot very similar to the knot in Gross-Pitaevskii theory. Consider a ‘‘charged’’ two-component BEC described by a complex doublet ϕ interacting ‘‘electromagnetically’’, which can be described by the following non-relativistic gauged Gross-Pitaevskii type Lagrangian (we will discuss the relativistic generalization later)

$$\mathcal{L} = i\frac{\hbar}{2} \left[\phi^\dagger (\tilde{D}_t \phi) - (\tilde{D}_t \phi)^\dagger \phi \right] - \frac{\hbar^2}{2m} |\tilde{D}_i \phi|^2 + \mu^2 \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2 - \frac{1}{4} \tilde{F}_{\mu\nu}^2, \quad (12)$$

where $\tilde{D}_\mu \phi = (\partial_\mu - ig\tilde{A}_\mu)\phi$, and μ^2 and λ are the chemical potential and the quartic coupling constant. Normalizing ϕ to $(\sqrt{2m/\hbar})\phi$ and putting

$$\phi = \frac{1}{\sqrt{2}} \rho \xi, \quad (\xi^\dagger \xi = 1) \quad (13)$$

we have the following Hamiltonian in the static limit,

$$\mathcal{H} = \frac{1}{2} (\partial_i \rho)^2 + \frac{\rho^2}{2} |\tilde{D}_i \xi|^2 - \frac{\mu^2}{2} \rho^2 + \frac{\lambda}{8} \rho^4 + \frac{1}{4} \tilde{F}_{ij}^2. \quad (14)$$

Notice that here we have also normalized μ^2 and λ to $(\hbar^2/2m)\mu^2$ and $(\hbar^2/2m)^2\lambda$. From now on we will use the normalized Hamiltonian (14).

At this point it is important to realize that the ‘‘electromagnetic’’ interaction here should be self-induced, since we are dealing with neutral condensates. So we identify the ‘‘electromagnetic’’ potential by the velocity field of the doublet ξ ,

$$\tilde{A}_\mu = -\frac{i}{g}\xi^\dagger\partial_\mu\xi. \quad (15)$$

A justification for this is that the $U(1)$ gauge invariance almost dictates us to identify the velocity field as the gauge potential. Indeed in the absence of the Maxwell term, (15) becomes an equation of motion. With this identification the field strength becomes the vorticity field

$$\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu = -\frac{i}{g}(\partial_\mu\xi^\dagger\partial_\nu\xi - \partial_\nu\xi^\dagger\partial_\mu\xi). \quad (16)$$

One might wonder why we have included the vorticity interaction in the Lagrangian (12), when this is absent in the Gross-Pitaevskii Lagrangian. The reason is because it costs energy to create the vorticity in two-component BEC. In ordinary BEC one does not have to worry about the vorticity because the vorticity is identically zero (since the velocity is given by the gradient of the phase of the one-component complex condensate). But a non-Abelian (multi-component) BEC has a non-vanishing vorticity, in which case it is natural to include the vorticity interaction in the Lagrangian [15, 16].

With (15) the Hamiltonian (14) becomes

$$\begin{aligned} \mathcal{H} = & \frac{1}{2}(\partial_i\rho)^2 + \frac{\rho^2}{2}\left(|\partial_i\xi|^2 - |\xi^\dagger\partial_i\xi|^2\right) - \frac{\lambda}{8}(\rho^2 - \rho_0^2)^2 \\ & - \frac{1}{4g^2}(\partial_i\xi^\dagger\partial_j\xi - \partial_j\xi^\dagger\partial_i\xi)^2, \\ \rho_0^2 = & \frac{2\mu^2}{\lambda}. \end{aligned} \quad (17)$$

Notice that now the coupling constant g represents the coupling strength of the vorticity. Minimizing the Hamiltonian we have the following equation of motion

$$\begin{aligned} \partial^2\rho - \left(|\partial_i\xi|^2 - |\xi^\dagger\partial_i\xi|^2\right)\rho = & \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho, \\ \left\{(\partial^2 - \xi^\dagger\partial^2\xi) + 2\left(\frac{\partial_i\rho}{\rho} - \xi^\dagger\partial_i\xi\right)(\partial_i - \xi^\dagger\partial_i\xi)\right. \\ & \left. - \frac{1}{g^2\rho^2}\left[\partial_i(\partial_i\xi^\dagger\partial_j\xi - \partial_j\xi^\dagger\partial_i\xi)\right](\partial_j - \xi^\dagger\partial_j\xi)\right\}\xi = 0. \end{aligned} \quad (18)$$

To understand the meaning of this notice that with

$$\hat{n} = \xi^\dagger\vec{\sigma}\xi, \quad (19)$$

we have

$$\begin{aligned} |\partial_\mu \xi|^2 - |\xi^\dagger \partial_\mu \xi|^2 &= \frac{1}{4}(\partial_\mu \hat{n})^2, \\ i(\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi) &= \frac{1}{2} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \frac{1}{2} N_{\mu\nu}, \end{aligned} \quad (20)$$

where $N_{\mu\nu}$ is mathematically identical to what we have in Skyrme theory in (7). This tells that the potential C_μ for the two-form $N_{\mu\nu}$ in (7) is given by (up to a gauge transformation)

$$C_\mu = 2g\tilde{A}_\mu = -2i\xi^\dagger \partial_\mu \xi. \quad (21)$$

More importantly, with (20) the Hamiltonian (17) can be expressed as

$$\mathcal{H} = \frac{1}{2}(\partial_i \rho)^2 + \frac{\rho^2}{8}(\partial_i \hat{n})^2 + \frac{\lambda}{8}(\rho^2 - \rho_0^2)^2 + \frac{1}{16g^2}(\partial_i \hat{n} \times \partial_j \hat{n})^2. \quad (22)$$

So the theory becomes very similar to Skyrme-Faddeev theory, which strongly indicates that the gauge theory of two-component BEC can allow a knot. As importantly this strongly implies that the Skyrme-Faddeev theory could play an important role in condensed matter physics.

To simplify the equation (18) notice that from (15) and (19) we have

$$\partial_\mu \xi = (ig\tilde{A}_\mu + \frac{1}{2}\vec{\sigma} \cdot \partial_\mu \hat{n})\xi. \quad (23)$$

With the identity the second equation of (18) is reduced to

$$\begin{aligned} (A + \vec{B} \cdot \vec{\sigma})\xi &= 0, \\ A &= (\partial_i \hat{n})^2, \\ \vec{B} &= \partial^2 \hat{n} + 2\frac{\partial_i \rho}{\rho} \partial_i \hat{n} - \frac{i}{2g^2 \rho^2} \partial_i N_{ij} \partial_j \hat{n}, \end{aligned} \quad (24)$$

which can be written as

$$A + \vec{B} \cdot \hat{n} = 0, \quad \hat{n} \times \vec{B} - i\hat{n} \times (\hat{n} \times \vec{B}) = 0, \quad (25)$$

or

$$\hat{n} \times (\partial^2 \hat{n} + 2\frac{\partial_i \rho}{\rho} \partial_i \hat{n}) + \frac{1}{2g^2 \rho^2} \partial_i N_{ij} \partial_j \hat{n} = 0. \quad (26)$$

So we can put (18) into the form

$$\begin{aligned} \partial^2 \rho - \frac{1}{4}(\partial_i \hat{n})^2 \rho &= \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho, \\ \hat{n} \times \partial^2 \hat{n} + 2\frac{\partial_i \rho}{\rho} \hat{n} \times \partial_i \hat{n} + \frac{1}{g^2 \rho^2} \partial_i N_{ij} \partial_j \hat{n} &= 0. \end{aligned} \quad (27)$$

This is the equation of two-component BEC that we are looking for. The similarity between this and the equation (7) of Skyrme-Faddeev theory is unmistakable.

Notice that the target space of ξ and \hat{n} is S^3 and S^2 , but here we have transformed the equation for ξ in (18) completely into the equation for \hat{n} in (27). This is made possible because, with the Abelian gauge invariance of (6), the physical target space of ξ becomes the gauge orbit space $S^2 = S^3/S^1$ which forms a CP^1 space. This means that we can view the theory as a self interacting CP^1 theory (coupled to a scalar field ρ), and replace ξ completely in terms of \hat{n} .

To show that the theory has a knot solution we construct a helical vortex solution first. To do this we choose the ansatz

$$\begin{aligned} \rho &= \rho(\rho), \quad \xi = \begin{pmatrix} \cos \frac{f(\rho)}{2} \exp(-in\varphi) \\ \sin \frac{f(\rho)}{2} \exp(ikz) \end{pmatrix}, \\ \hat{n} &= \xi^\dagger \vec{\sigma} \xi = \begin{pmatrix} \sin f(\rho) \cos(mkz + n\varphi) \\ \sin f(\rho) \sin(mkz + n\varphi) \\ \cos f(\rho) \end{pmatrix}. \end{aligned} \quad (28)$$

With this (27) is reduced to

$$\begin{aligned} \ddot{\rho} + \frac{1}{\rho} \dot{\rho} - \frac{1}{4} \left(\dot{f}^2 + (m^2 k^2 + \frac{n^2}{\rho^2}) \sin^2 f \right) \rho &= \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho, \\ \left(1 + (m^2 k^2 + \frac{n^2}{\rho^2}) \frac{\sin^2 f}{g^2 \rho^2} \right) \ddot{f} + \left(\frac{1}{\rho} + 2 \frac{\dot{\rho}}{\rho} + (m^2 k^2 + \frac{n^2}{\rho^2}) \frac{\sin f \cos f}{g^2 \rho^2} \right) \dot{f} \\ + \frac{1}{\rho} \left(m^2 k^2 - \frac{n^2}{\rho^2} \right) \frac{\sin^2 f}{g^2 \rho^2} \dot{f} - (m^2 k^2 + \frac{n^2}{\rho^2}) \sin f \cos f &= 0. \end{aligned} \quad (29)$$

So with the boundary condition

$$\dot{\rho}(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = \pi, \quad f(\infty) = 0, \quad (30)$$

we obtain the non-Abelian vortex solutions shown in Fig.2.

There are three points that have to be emphasized here. First, when $m = 0$, the solution describes the untwisted non-Abelian vortex [15]. But when m is not zero, it describes a helical vortex which is periodic in z -coordinate. In this case, the vortex has a non-vanishing velocity current (not only around the vortex but also) along the z -axis. Secondly, the doublet ϕ starts from the second component at the core, but the first component takes over completely at the infinity. This is due to the boundary condition $f(0) = \pi$ and $f(\infty) = 0$, which assures that our solution describes a genuine non-Abelian vortex. Thirdly, when $f = 0$ (or $f = \pi$) the doublet effectively becomes a singlet, and (29) describes the well-known vortex in single-component BEC. Only when f has a non-trivial profile, we have a non-Abelian vortex.

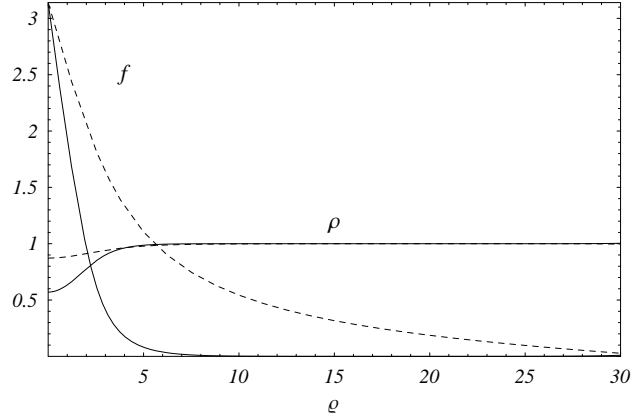


Figure 2: The non-Abelian vortex (dashed line) with $m = 0, n = 1$ and the helical vortex (solid line) with $m = n = 1$ in the gauge theory of two-component BEC. Here we have put $\lambda/g^2 = 1$, $k = 0.64 \sqrt{\lambda} \rho_0$, and ρ is in the unit of $1/\sqrt{\lambda} \rho_0$.

In Skyrme theory the helical vortex is interpreted as a twisted magnetic vortex whose flux is quantized [6, 14] Now we show that the above vortex is a twisted vorticity flux which is also quantized. To see this notice that the non-Abelian structure of the vortex is represented by the doublet ξ . Moreover, the velocity field of the doublet is given by

$$\tilde{A}_\mu = -\frac{n}{2g}(\cos f + 1)\partial_\mu \varphi - \frac{mk}{2g}(\cos f - 1)\partial_\mu z, \quad (31)$$

which generates the vorticity

$$\begin{aligned} H_{\mu\nu} &= -\frac{i}{g}(\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi) \\ &= \frac{\dot{f}}{2g} \sin f \left(n(\partial_\mu \rho \partial_\nu \varphi - \partial_\nu \rho \partial_\mu \varphi) + mk(\partial_\mu \rho \partial_\nu z - \partial_\nu \rho \partial_\mu z) \right). \end{aligned} \quad (32)$$

This has two vorticity fluxes, $\phi_{\hat{z}}$ along the z -axis

$$\phi_{\hat{z}} = \int H_{\hat{\rho}\hat{\phi}} \rho d\rho d\varphi = -\frac{2\pi n}{g}, \quad (33)$$

and $\phi_{\hat{\phi}}$ around the the z -axis (in one period section from $z = 0$ to $z = 2\pi/k$)

$$\phi_{\hat{\phi}} = \int_{z=0}^{z=2\pi/k} H_{\hat{z}\hat{\rho}} d\rho dz = \frac{2\pi m}{g}. \quad (34)$$

This shows that the helical vortex is made of two quantized vorticity fluxes, the $\phi_{\hat{z}}$ -flux which is concentrated at the core and the $\phi_{\hat{\phi}}$ -flux which surrounds it. This confirms that

the helical vortex is a twisted vorticity flux which is very similar to the helical vortex in Gross-Pitaevskii theory [16].

Now, with the helical vortex, one can easily make a twisted vortex ring smoothly connecting the periodic ends together. And just as in Skyrme theory the twisted vortex ring becomes a topological knot. But here it is $\pi_3(S^3)$ of the doublet ξ which provides the non-trivial quantum number,

$$q = -\frac{1}{4\pi^2} \int \epsilon_{ijk} \xi^\dagger \partial_i \xi (\partial_j \xi^\dagger \partial_k \xi) d^3x. \quad (35)$$

Of course this is identical to the expression (11), due to the Hopf fibering of S^3 to $S^2 \times S^1$ [16]. This tells that we can express the knot quantum number either by $\pi_3(S^3)$ or by $\pi_3(S^2)$.

Obviously this knot has a topological stability, because the knot topology (11) can not be changed by a smooth deformation of the field configuration. Moreover it has a dynamical stability. To understand this notice that the knot has a twisted velocity field so that it has a non-vanishing velocity around the z -axis. This means that it carries a non-vanishing angular momentum along the z -axis. And this angular momentum provides the dynamical stability, because it creates a centrifugal force that prevents the collapse of the knot. Notice that this dynamical stability originates from the knot topology, because the angular momentum comes from the twisted velocity field. In this sense the topological stability and the dynamical stability have one and the same origin. It is this remarkable interplay between topology and dynamics which assures the stability of the knot. The nontrivial twisted topology of the knot expresses itself in the form of the angular momentum, which in turn provides the dynamical stability of the knot. This presence of the angular momentum is what differentiates our knot from the untwisted Abrikosov-type vortex ring which has no dynamical stability.

There have been assertions that two-component BEC admits a knot [10]. But notice that this knot is based on Gross-Pitaevskii theory of two-component BEC, which has no vorticity interaction. In contrast our knot is based on the gauge theory in which the vorticity interaction plays a crucial role. Nevertheless physically two knots are very similar [16]. Both can be identified as a vorticity knot. This implies that both theories should be taken seriously as a theory of two-component BEC.

4. Knot in Two-gap Superconductors

In the above gauge theory of spin-1/2 condensates the gauge interaction was a self-induced interaction. But when the doublet is charged, the gauge interaction can be treated as independent. In this case the theory can describe a two-gap superconductor. But even in this case the knot topology and thus the knot itself should survive. This implies that two-gap superconductor should also have a topological knot.

The knot in two-gap superconductor could be either relativistic or non-relativistic, and appear in both Abelian and non-Abelian setting [17]. In this paper we will discuss the relativistic knot in the Abelian setting (a non-relativistic Gross-Pitaevskii type theory

gives an identical result). Consider a charged doublet scalar field ϕ coupled to the real electromagnetic field,

$$\begin{aligned}\mathcal{L} &= -|D_\mu\phi|^2 + \mu^2\phi^\dagger\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}^2, \\ D_\mu\phi &= (\partial_\mu - igA_\mu)\phi.\end{aligned}\quad (36)$$

The Lagrangian has the equation of motion

$$\begin{aligned}D^2\phi &= \lambda(\phi^\dagger\phi - \frac{\mu^2}{\lambda})\phi, \\ \partial_\mu F_{\mu\nu} &= j_\nu = ig[(D_\nu\phi)^\dagger\phi - \phi^\dagger(D_\nu\phi)].\end{aligned}\quad (37)$$

Now, with

$$\phi = \frac{1}{\sqrt{2}}\rho\xi, \quad \xi^\dagger\xi = 1, \quad \hat{n} = \xi^\dagger\vec{\sigma}\xi, \quad (38)$$

we can reduce (37) to [17]

$$\begin{aligned}\partial^2\rho - \left(\frac{1}{4}(\partial_\mu\hat{n})^2 + g^2(A_\mu + \tilde{A}_\mu)^2\right)\rho &= \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho, \\ \hat{n} \times \partial^2\hat{n} + 2\frac{\partial_\mu\rho}{\rho}\hat{n} \times \partial_\mu\hat{n} + \frac{2}{g\rho^2}\partial_\mu F_{\mu\nu}\partial_\nu\hat{n} &= 0, \\ \partial_\mu F_{\mu\nu} &= j_\mu = g^2\rho^2(A_\mu + \tilde{A}_\mu), \\ \tilde{A}_\mu &= -\frac{i}{g}\xi^\dagger\partial_\mu\xi, \quad \rho_0 = \frac{2\mu^2}{\lambda}.\end{aligned}\quad (39)$$

This is the equation for two-gap superconductor. Notice that with $A_\mu = -\tilde{A}_\mu$ the first two equations reduce to (27). This tells that the gauge theory of two-component BEC and the above theory of two-gap superconductor are closely related.

To obtain the desired knot we first construct a superconducting helical magnetic vortex. Let

$$\begin{aligned}\rho &= \rho(\rho), \quad \xi = \begin{pmatrix} \cos\frac{f(\rho)}{2}\exp(-in\varphi) \\ \sin\frac{f(\rho)}{2}\exp(imkz) \end{pmatrix}, \\ A_\mu &= \frac{1}{g}(nA_1(\rho)\partial_\mu\varphi + mkA_2(\rho)\partial_\mu z), \\ \hat{n} &= \xi^\dagger\vec{\sigma}\xi = \begin{pmatrix} \sin f(\rho)\cos(n\varphi + mkz) \\ \sin f(\rho)\sin(n\varphi + mkz) \\ \cos f(\rho) \end{pmatrix}, \\ \tilde{A}_\mu &= -\frac{n}{2g}(\cos f(\rho) + 1)\partial_\mu\varphi - \frac{mk}{2g}(\cos f(\rho) - 1)\partial_\mu z.\end{aligned}\quad (40)$$

With this we have

$$j_\mu = g\rho^2 \left(n \left(A_1 - \frac{\cos f + 1}{2} \right) \partial_\mu \varphi + mk \left(A_2 - \frac{\cos f - 1}{2} \right) \partial_\mu z \right), \quad (41)$$

and (39) becomes

$$\begin{aligned} & \ddot{\rho} + \frac{1}{\rho} \dot{\rho} - \left[\frac{1}{4} \left(\dot{f}^2 + \left(\frac{n^2}{\rho^2} + m^2 k^2 \right) \sin^2 f \right) \right. \\ & \left. + \frac{n^2}{\rho^2} \left(A_1 - \frac{\cos f + 1}{2} \right)^2 + m^2 k^2 \left(A_2 - \frac{\cos f - 1}{2} \right)^2 \right] \rho \\ & = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho, \\ & \ddot{f} + \left(\frac{1}{\rho} + 2 \frac{\dot{\rho}}{\rho} \right) \dot{f} - 2 \left(\frac{n^2}{\rho^2} \left(A_1 - \frac{1}{2} \right) + m^2 k^2 \left(A_2 + \frac{1}{2} \right) \right) \sin f = 0, \\ & \ddot{A}_1 - \frac{1}{\rho} \dot{A}_1 - g^2 \rho^2 \left(A_1 - \frac{\cos f + 1}{2} \right) = 0, \\ & \ddot{A}_2 + \frac{1}{\rho} \dot{A}_2 - g^2 \rho^2 \left(A_2 - \frac{\cos f - 1}{2} \right) = 0. \end{aligned} \quad (42)$$

Now, we impose the following boundary condition for the non-Abelian vortices [17],

$$\begin{aligned} \rho(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = \pi, \quad f(\infty) = 0, \\ A_1(0) = -1, \quad A_1(\infty) = 1. \end{aligned} \quad (43)$$

This need some explanation, because the boundary value $A_1(0)$ is chosen to be -1 , not 0 . This is to assure the smoothness of $\rho(\rho)$ and $f(\rho)$ at the origin. Only with this boundary value they become analytic at the origin. At this point one might object the boundary condition, because it creates an apparent singularity in the gauge potential at the origin. But notice that this singularity is an unphysical (coordinate) singularity which can easily be removed by a gauge transformation. In fact the singularity disappears with the gauge transformation

$$\phi \rightarrow \phi \exp(in\varphi), \quad A_\mu \rightarrow A_\mu + \frac{n}{g} \partial_\mu \varphi, \quad (44)$$

which changes the boundary condition $A_1(0) = -1$ and $A_1(\infty) = 1$ to $A_1(0) = 0$ and $A_1(\infty) = 2$. Mathematically this boundary condition has a deep origin, which has to do with the fact that the Abelian $U(1)$ runs from 0 to 2π , but the S^1 fiber of $SU(2)$ runs from 0 to 4π [17]. As for $A_2(\rho)$, we choose $A_2(\infty) = 0$ to make the supercurrent vanishing at infinity and require the vortex superconducting. As we will see, this requires a logarithmic divergence for $A_2(0)$. The boundary condition will have an important consequence in the following.

With the boundary condition we can integrate (42) and obtain the non-Abelian vortex solution of the two-gap superconductor, which is shown in Fig.3. The solution

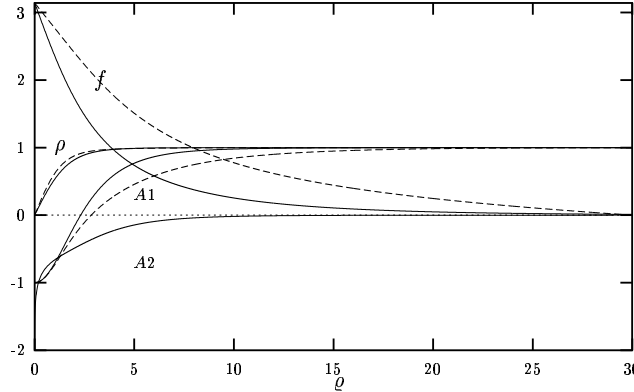


Figure 3: The non-Abelian vortex (dashed line) with $m = 0, n = 1$ and the helical vortex (solid line) with $m = n = 1$ in two-gap superconductor. Here we have put $g = 1$, $\lambda = 2$, $k = \rho_0/10$, and ρ is in the unit of $1/\rho_0$. Notice that A_2 has a logarithmic singularity at the origin.

is very similar to the one we have in two-component BEC. When $m = 0$, the solution (with $A_2 = 0$) describes an untwisted non-Abelian vortex [17]. But when m is not zero, it describes a helical magnetic vortex which is periodic in z -coordinate. Moreover, the vortex starts from the second component at the core, but the first component takes over completely at the infinity. This is due to the boundary condition $f(0) = \pi$ and $f(\infty) = 0$, which assures that our solution describes a genuine non-Abelian vortex. This is true even when $m = 0$. Only when $f = 0$ (or $f = \pi$) the doublet effectively becomes a singlet, and (42) describes the Abelian Abrikosov vortex of one-gap superconductor.

There are important differences between the non-Abelian vortex and the Abrikosov vortex. First the non-Abelian vortex has a non-Abelian magnetic flux quantization [17]. Indeed the quantized magnetic flux $\hat{\phi}_z$ of the non-Abelian vortex along the z -axis is given by

$$H_z = \frac{n \dot{A}_1}{g \rho},$$

$$\hat{\phi}_z = \int H_z d^2x = \frac{2\pi n}{g} [A_1(\infty) - A_1(0)] = \frac{4\pi n}{g}. \quad (45)$$

Notice that the unit of the non-Abelian flux is $4\pi/g$, not $2\pi/g$. This is a direct consequence of the boundary condition (43). This non-Abelian quantization of magnetic flux comes from the non-Abelian topology $\pi_2(S^2)$ of the doublet ξ , or equivalently the triplet \hat{n} , whose topological quantum number is given by [17]

$$q = \frac{g}{4\pi} \int H_z d^2x = -\frac{1}{4\pi} \int \epsilon_{ij} \partial_i \xi^\dagger \partial_j \xi d^2x$$

$$= \frac{1}{8\pi} \int \epsilon_{ij} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}) d^2x = n. \quad (46)$$

This distinguishes our non-Abelian vortex from the Abelian vortex whose topology is fixed by $\pi_1(S^1)$.

Another important feature of the non-Abelian vortex is that it carries a non-vanishing supercurrent along the z -axis,

$$\begin{aligned} i_z &= mkg \int \rho^2 \left(A_2 - \frac{\cos f + 1}{2} \right) \rho d\rho d\phi \\ &= \frac{2\pi mk}{g} \int \left(\ddot{A}_2 + \frac{1}{\rho} \dot{A}_2 \right) \rho d\rho = \frac{2\pi mk}{g} (\rho \dot{A}_2) \Big|_{\rho=0}^{\rho=\infty} = -\frac{2\pi mk}{g} (\rho \dot{A}_2) \Big|_{\rho=0}. \end{aligned} \quad (47)$$

This is due to the logarithmic divergence of A_2 at the origin.

Notice that the superconducting helical vortex has only a heuristic value, because one needs an infinite energy to create it (since the magnetic flux around the vortex becomes divergent because of the singularity of A_2 at the origin). With the helical vortex, however, one can make a vortex ring by smoothly bending and connecting two periodic ends. In the vortex ring the infinite magnetic flux of A_2 can be made finite making the finite supercurrent (47) of the vortex ring produce a finite flux, and we can fix the flux to have the value $4\pi m/g$ by adjusting the current with k . With this the vortex ring now becomes a topologically stable knot.

To see this notice that the doublet ξ , after forming a knot, acquires a non-trivial topology $\pi_3(S^2)$ which provides the knot quantum number,

$$\begin{aligned} Q &= -\frac{1}{4\pi^2} \int \varepsilon_{ijk} \xi^\dagger \partial_i \xi (\partial_j \xi^\dagger \partial_k \xi) d^3x \\ &= \frac{g^2}{32\pi^2} \int \varepsilon_{ijk} C_i (\partial_j C_k - \partial_k C_j) d^3x = mn. \end{aligned} \quad (48)$$

This is nothing but the Chern-Simon index of the potential C_μ , which is mathematically identical to the quantum number of the knots we discussed before. This tells that our knot is also made of two quantized magnetic flux rings linked together whose knot quantum number is fixed by the linking number mn . Obviously two flux rings linked together can not be separated by any continuous deformation of the field configuration. This provides the topological stability of the knot.

Again this topological stability is backed up by a dynamical stability. To see this notice that the supercurrent of the knot has two components, the one around the knot tube which confines the magnetic flux along the knot, but more importantly the other along the knot which creates a magnetic flux passing through the knot disk. This component of supercurrent along the knot now generates a net angular momentum which provides the centrifugal repulsive force preventing the knot to collapse. This makes the knot dynamically stable.

To compare our knot with the Abrikosov vortex ring (made of the Abrikosov vortex in conventional superconductor), notice that the Abrikosov knot is empty (i.e., does not carry a net supercurrent). As importantly it is unstable, and collapses immediately. In contrast our knot has a helical supercurrent, and is stable. Furthermore these two features are deeply related. The helical supercurrent plays a crucial role to stabilize the

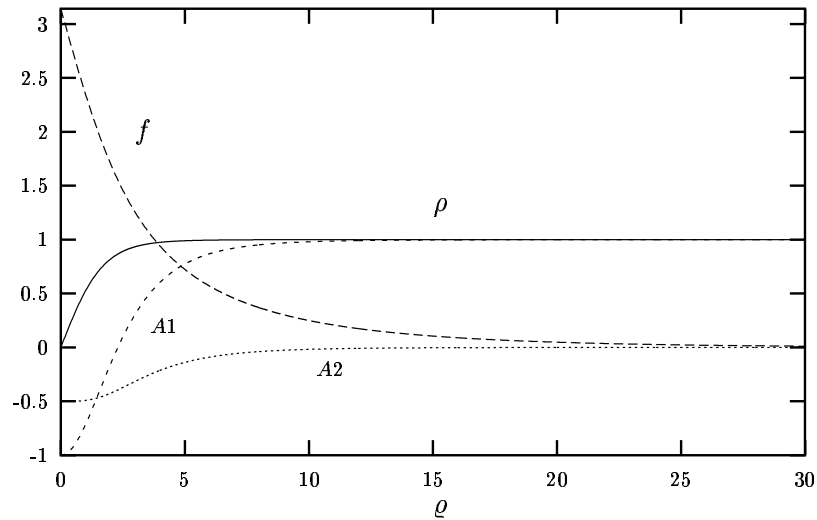


Figure 4: The regular helical magnetic vortex with $m = n = 1$ in two-gap superconductor. Here we have put $g = 1$, $\lambda = 2$, $k = 0.12\rho_0$, $A_2(0) = 0.5$, and ρ is in the unit of $1/\rho_0$. Notice that the solution is completely regular.

vortex ring by providing the net angular momentum, which prevents the collapse of the vortex ring. And this helical supercurrent originates from the knot topology. This remarkable interplay between topology and dynamics is what provides the stability of the knot. The nontrivial topology expresses itself in the form of the helical supercurrent, which in turn provides the dynamical stability of the knot. We emphasize that this supercurrent is what distinguishes our knot from the Abrikosov vortex ring, which has neither topological nor dynamical stability.

5. Discussion

In this paper we have presented a compelling argument for the existence of topological knots in two-component BEC and two-gap superconductor. Similar knots have popped out almost everywhere, in particular in high energy physics in QCD [18] and Weinberg and Salam model [19]. But we emphasize that at the center of these topological objects lies the baby skyrmion and the Faddeev-Niemi knot [6, 14]. In fact, our helical vortices and knots in this paper are a straightforward generalization of the baby skyrmion and the Faddeev-Niemi knot.

It has been assumed that the topological objects in Skyrme theory can only be realized at high energy, at the hadronic scale. But our analysis shows that similar objects could exist in a completely different environment, at a much lower scale, in low energy condensed matters. If so, the challenge now is to verify the existence of the topological knot experimentally in condensed matters. Constructing the knot might not be so easy at present moment. Nevertheless, with some experimental ingenuity, one should be able

to construct the knots in condensed matters.

Note Added: One might doubt the existence of a superconducting knot because the superconducting helical vortex we discussed in Section IV was singular (and thus unphysical). In this note we report a regular superconducting helical vortex which has a finite magnetic flux around the axis and thus a finite energy. The regular solution is obtained linking $A_2(0)$ with k . For example for $k = 0.12$ we obtain the regular solution shown in Fig.4, with $A_2(0) = 0.5$. This type of regular helical vortex has vanishing supercurrent i_z , but could still be called superconducting because it has a non-trivial supercurrent density j_z which generates a net magnetic flux H_ϕ around the vortex. The regular helical vortex strongly support the existence of a regular knot. The details will be published elsewhere.

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