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Crossover Effect on the Susceptibility Amplitude Ratio Using Modified Gaussian Model

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Abstract

The susceptibility amplitude ratio with uniaxial dipolar is calculated using modified Gaussian model. Incorporating the renormalization group (RG) theory, the crossover behavior of the calculated ratio to the pure Gaussian isotropic point is discussed in terms of the crossover parameters. A comparison between our calculations and the available theoretical and experimental results is presented.

Introduction

Amplitude ratios[1-11] for different physical quantities, such as specific heat and susceptibility near a critical point, are considered a crucial quantity to characterize the universality of the thermodynamic behavior. Many theoretical and experimental work has been done on the universal combinations of critical amplitude ratio at the critical point, but a few have been done in the crossover region[7,8]. This has motivated us to study the crossover behavior of the susceptibility amplitude ratio for the cases of uniaxial dipolar ferromagnetes[12] (reference 12 shall be quoted many times and shall be denoted as article I1 throughout our paper). The results of these calculations may be helpful in understanding and explaining the recent experimental results in systems that may have this kind of behavior[4-6].

Including the crossover's behavior in the ratio calculation will enrich our understanding in characterizing the critical phenomena. Theoretical study of such behavior could be treated using different approaches. Lately, two extensive review articles[2,3] discussed the crossovers, in different systems, in the frame of the renormalization group theory. Their articles will allow us to retrieve our, and related, work. For example, the crossover of the specific heat amplitude ratio for anisotropic[7] and uniaxial[8] dipolar interaction of m-type Lifshitz critical behavior is investigated using the Gaussian model. Also, We applied the RG theory[10] to calculate the specific heat and susceptibility amplitude ratio in the neighborhood of m-Lifshitz point, to one-loop order.

In this paper we consider the susceptibility amplitude ratio in the neighborhood of uniaxial dipolar ferromagnets as discussed in I1. The ratio will include the crossover behavior to the Gaussian usual point. Due to a mathematical difficulty regarding the singularity in the Gaussian integrals, it will not be straight forward to use the Gaussian model directly. Therefore, we are going to propose a new approach, which we call ‘Modified Gaussian Model’. In this new approach, we will incorporate the RG theory, to one loop order, to modify our previous work[7,8] of the Gaussian model. It is worth mentioning that a comprehensive solution to the given problem can only be obtained within the framework of the renormalization-group theory to one[13] and two-loop orders[14], if it is possible.

The text is organized as follows: In section II, we will retrieve the simple model of the uniaxial dipolar ferromagnets (I1). The results of section II will be used as a corner stone in calculating the ratio using a simplified version of the Gaussian model in section III. The flow of the coupling constants and the ratio using the modified Gaussian model will be calculated in section IV. A discussion will be given in section V, followed by a short summary of the results and a conclusion of the work in section VI. Important analytical expressions that are used in our calculation are given in Appendix A. Through our paper, we will use the subscripts + and – for the values of the susceptibility above and below the critical temperature T_c, respectively.

Uniaxial Dipolar Ferromagnets

In this section we will briefly review the calculation of the susceptibility[15] χ (which measures the response to an external field, for example magnetic field (B) in general. Let us consider the partition function

$$Z = \int_{\varphi} e^{-H(\varphi)} \tag{1}$$

where $H(\varphi)$ is the Landau-Ginzburg-Wilson free-energy functional (see I1). In our work, $H(\varphi)$ describes the critical behavior of a system that exhibits uniaxial dipolar interactions and can be written in the form

$$\begin{aligned} H = & -\frac{1}{2}\int_{k}(r_o + p^2 + q^2 + g^2\frac{q^2}{p^2})\varphi_o(\mathbf{k})\varphi_o(-\mathbf{k}) \\ & -\frac{u_o}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\varphi_o(\mathbf{k_1})\varphi_o(\mathbf{k_2})\varphi_o(\mathbf{k_2})\varphi_o(\mathbf{k_4})\times\delta\left[\sum_{i=1}^{4}\mathbf{k_i}\right], \end{aligned} \tag{2}$$

where $r_o = (T - T_c)/T_c$ is the bare reduced temperature, u_o is the coupling constant, and they are unrenormalized. The dipolar coupling constant g contains the limits of the usual behavior (at $g = 0$) and the uniaxial dipolar behavior (at $g > 0$). Here $\varphi_o(\mathbf{k})$ represents the order parameter, e.g., magnetization, polarization, etc. The d-dimension wave vector $\mathbf{k} = (\mathbf{p}, \mathbf{q})$ is decomposed into $\mathbf{q}$, the component along the uniaxial direction, and $\mathbf{p}$, the remaining (d-1) components. In equation (2) we used the notation $\int_k = \frac{1}{(2\pi)^d}\int d^d k$, and the Gaussian propagator for the graphical expansion can be written as:

$$G(r_o, g) = (r_o + p^2 + q^2 + g^2 \frac{q^2}{p^2})^{-1}. \tag{3}$$

The susceptibility is defined in terms of free energy per unit volume F and is related to Z by the relation $F = -\frac{1}{V}\ln Z$. Then the susceptibility is given by

$$\chi = \frac{\partial M}{\partial B} = -\frac{1}{\beta}\frac{\partial^2 F}{\partial B^2}\bigg|_{B=0} \propto \tilde{G}_c^{(2)} \propto 1/\hat{\Gamma}^{(2,0)}, \tag{4}$$

where M denotes the magnetization (i.e. the total magnetic moment), $\beta = 1/T$, $\tilde{G}_c^{(2)} = \langle \varphi^2 \rangle - \langle \varphi \rangle^2$ is the two-point correlation function and $\hat{\Gamma}^{(2,0)}$ is the two-point vertex function.

Above the critical temperature T_c, it was found that (see I1) the singular part of the susceptibility reads:

$$\chi_+^{-1} = r_+^{\gamma}\hat{\Gamma}_{+R}^{(2,0)}(r, g) \propto \left[r + \tfrac{u}{2} I_1(r, g) \right]. \tag{5}$$

Below, the critical temperature T_c, it was found that the singular part of the susceptibility[10] reads:

$$\chi_-^{-1} = r_-^{\gamma}\hat{\Gamma}_{-R}^{(2,0)}(r_-, g) \propto \left[r_- + uI_1(r_-, g) + \tfrac{3}{2} r_- \, uI_2(r_-, g) \right], \tag{6}$$

where $r_- \to |-2r|$, γ is the effective exponent of susceptibility and the integrals $I_1(r, g)$ and $I_2(r, g)$ have the following form[8,16]:

$$I_1(r, g) = \int dq \int G(r, g) d^{d-1} p = \pi S_d I_c(r, g),$$

$$I_2(r, g) = \int dq \int G^2(r, g) d^{d-1} p = -\frac{\partial I_1(r, g)}{\partial r},$$

where $S_d^{-1} = \frac{2}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}$, ($d > 1$) is the geometrical factor of integration, and

$$I_c(r, g) = \int \frac{d^d p}{\sqrt{r + p^2}\sqrt{g^2 + p^2}} = \frac{1}{2}\Gamma(\frac{d}{2})\Gamma(\alpha)(\frac{g + \sqrt{r}}{2})^{d-2} \; {}_2F_1(\alpha, \alpha; 1; W), \tag{7}$$

with $\Gamma(\delta)$ is the usual gamma function and $\alpha = 1 - \frac{d}{2}$, $W = (\frac{g-\sqrt{r}}{g+\sqrt{r}})^2$ and ${}_2F_1$ is the hypergeometric function. Equation (7) is the suitable analytic form, which gives the correct expression in both limits r = 0 or g = 0, i.e.

$$I_1(r=0,g) = \frac{\sqrt{\pi}}{4} S_d\, (g^2)^{d/2-1} \Gamma(\alpha)\, \Gamma(\frac{d-1}{2}),$$
$$I_1(r,g=0) = \frac{\sqrt{\pi}}{4} S_d\, (r)^{d/2-1} \Gamma(\alpha)\, \Gamma(\frac{d-1}{2}) \tag{8}$$

Simplified Gaussian Model

The Gaussian region[17] is defined as being far away from the critical point. In such a region, where one can use the Gaussian model, the fluctuations are kept only to the second order (the first part in equation 2) and neglecting the fourth order interaction of the fluctuations (which is the second part in equation 2). Because of the non-asymptotic behavior of this region, one does not expect the ratio to be universal, but it may depend on the non-universal parameters of the Gaussian model.

To calculate the susceptibility amplitude ratio in the Gaussian model, we will keep only the second term in equations 5 and 6. Thus, the susceptibility above and below T_c could be expressed as:

$$\chi_+^{-1} = \hat{\Gamma}_{+R}^{(2,0)} \propto I_1(r,g)/2,$$
$$\chi_-^{-1} = \hat{\Gamma}_{-R}^{(2,0)} \propto I_1(|-2r|,g) \tag{9}$$

The ratio is casting in the form:

$$\frac{\chi_+}{\chi_-} = \frac{I_1(|-2r|,g)}{I_1(r,g)/2} = 2\left[\frac{x+\sqrt{2}}{x+1}\right]^{d-2} R_f\,,$$
$$R_f = \frac{{}_2F_1(\alpha,\alpha;1;\left[(x-\sqrt{2})/(x+\sqrt{2})\right]^2)}{{}_2F_1(\alpha,\alpha;1;\left[(x-1)/(x+1)\right]^2)}, \tag{10}$$

where we introduced the "scaling variable" $x = g/\sqrt{r}$. The assumption in keeping only the term that contains the integral $I_1(r)$ has no solid physical criteria. It is used because it is simple; the pre-factor in the integral $I_1(r)$, which contains the singularity, and also the coupling constant u will be cancelled out.

The ratio (χ_+/χ_-) in equation (10) is displayed in Fig. 1. The solid line includes the ratio R_f and the dotted line excludes this ratio. It was found that the effect of the ratio of R_f in our calculation is insignificant. It cancels out at both limits and makes only a negligible contribution to the intermediate values of x (crossover region). As expected, at the pure Gaussian isotropic point, i.e. $x = 0$, the equation has the asymptotic expression:

$$\frac{\chi_+}{\chi_-} = 2^{d/2} = 2.83. \tag{11}$$

For $x > 0$, the ratio in equation (10) decreases monotonically in the crossover region, to the mean field value of 2 at $x \gg 0$, where the pure uniaxial dipolar point is dominant and responsible for decreasing the susceptibility ratio.

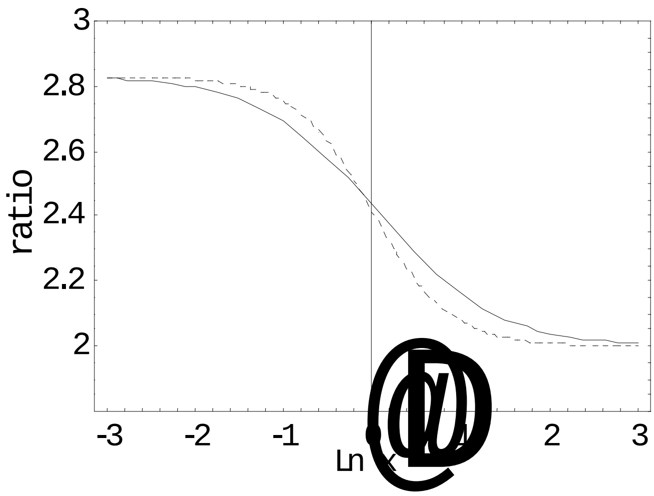

Figure 1: The susceptibility amplitude ratio in the simplified Gaussian model as a function of of Log[x],where $x = g/\sqrt{r}$ is the scaling variable. $x = 0$ and $x \to \infty$ correspond to the usual isotropic critical point and uniaxial dipole point, respectively. The solid line includes the ratio R_f , while the dotted line excludes this ratio.

Modified Gaussian Model

To merge the renormalization group theory into the previous section, let us recall the procedure of I1. In particular, after introducing the momentum scale μ , we have:

$$\begin{aligned} r_o &= Z_r r, \\ g_o &= Z_g g, \\ u_o &= \mu^{\varepsilon} Z_r u S_d^{-1}, \\ \varphi_o &= Z_{\varphi}^{\frac{1}{2}} \varphi, \end{aligned} \tag{12}$$

with $\varepsilon = 4 - d$ and the factor $S_d^{-1} = \dfrac{2}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}$ is introduced for convenience. In one loop order, the perturbation expansion parameters, e.g. Z_r and Z_u are given by:

$$Z_r = 1 + \frac{1}{2}\frac{u}{\varepsilon}\left[1 + g/\mu\right]^{-\varepsilon},$$
$$Z_u = 1 + \frac{3}{2}\frac{u}{\varepsilon}\left[1 + g/\mu\right]^{-\varepsilon} \quad (13)$$

The flow of the coupling constant $u(l)$, which is used in the two-point vertex function, can be determined through the numerical solution of the differential equation:

$$l\,\frac{du(l)}{dl} = \varepsilon\, u(l) + \frac{3}{2}u^2(l)\left[1 - \frac{1}{1 + \mu l / g(l)}\right]. \quad (14)$$

The solution of Eq. (14) is plotted in Fig. 2 as a function of the flow parameter l, and for different values of g. Note that the flow parameter plays[15] the role of a dummy variable which can be forgotten in the final results. The pronounce effect of small values of g can be realized and in the interval $(10^{-1} \geq l \geq 10^{-3})$, $u(l)$ starts to increase to the usual fixed-point value. At small l values $(l < 10^{-4})$, $u(l)$ starts to go back to the uniaxial fixed-point value (≈ 0.66).

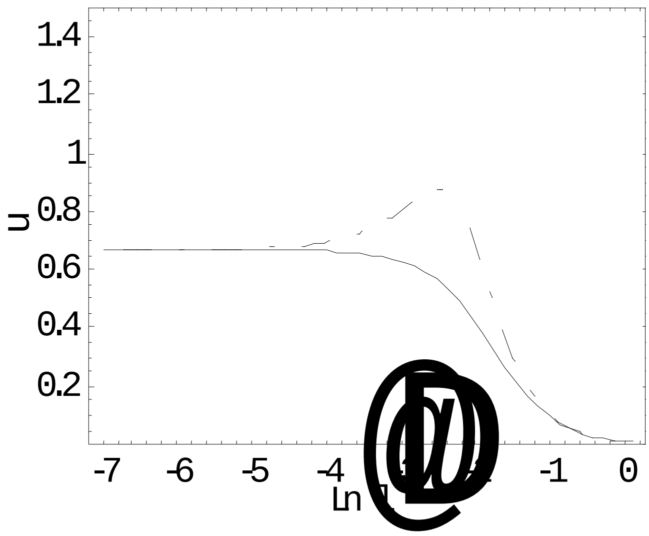

Fig. 2. The effective coupling $u(l)$ as a function of $\mathrm{Log}[l]$, l is the flow parameter, with $\mu = 1$, and for different values of g. The initial value $u(0) = 0.01$ has been taken. The solid line is for $g = 1.0$, while the dashed-dotted line is for $g = 0.01$.

Within the renormalization group theory and for $T > T_L$, the renormalized two-point vertex reads:

$$\hat{\Gamma}^{(2,0)}_{+R} = \left[r_o + \frac{u_o}{2} I_1(r_o) \right]$$
$$= \mu \left[Z_r \frac{r}{\mu^2} + \frac{u}{2} S_d^{-1} \left\{ J_1(\frac{r}{l^2\mu^2}, \frac{g}{l\mu}) - J_1(0, \frac{g}{l\mu}) \right\} \right]. \quad (15)$$

where the integral $J(r,g)$ is defined in appendix A. As a function of g and r we can have

$$\chi_+^{-1} = r_+^{\gamma} \hat{\Gamma}^{(2,0)}_{+R} = r_+^{\gamma} \left[1 - u(x+1)^{-\varepsilon} \left\{ 0.1 + 0.5x + 0.5x^2 \ln[\frac{x}{x+1}] \right\} \right]. \quad (16)$$

where

$$\gamma = 1 + \frac{1}{4} u(l) \left[1 + g / l\mu \right]^{-1} + \text{ negleted terms} \quad (17)$$

Following I1, we used in Equ. 16 the condition $\frac{r}{l^2\mu^2} = 1$ in calculating $\hat{\Gamma}^{(2,0)}_{R+}$ to get rid of the terms that have the form $\ln(\frac{r}{l^2\mu^2})$, then we used it again as $r = l^2\mu^2$ in the expression $x = g / l\mu = g / \sqrt{r}$.

At $T < T_L$, we will use Equation (6) and, keeping only the first two terms, the renormalized two-point vertex in the form gives:

$$\chi_-^{-1} = r_-^{\gamma} \hat{\Gamma}^{(2,0)}_{R-} = r_-^{\gamma} \left[r_- + uJ_1(r_-, \frac{g}{\mu}) \right]$$
$$= r_-^{\gamma} \left[1 + u(x+1)^{-\varepsilon-1} \left\{ 0.193 + x + x^2 + x^2 \ln[\frac{x}{x+1}] \right\} \right] \quad (18)$$

In equation 18, we used the condition $-\frac{2r}{l^2\mu^2} = 1$ in calculating $\hat{\Gamma}^{(2,0)}_{R-}$ to get rid of the terms that have the form $\ln(-\frac{2r}{l^2\mu^2})$, then we used it again as $|-2r| = l^2\mu^2$.

To facilitate comparison with the experiment[4-6], the effective exponent of the susceptibility is plotted in Fig. 3 as a function of scaling variable $\ln[x]$, $x = g / \sqrt{r}$.

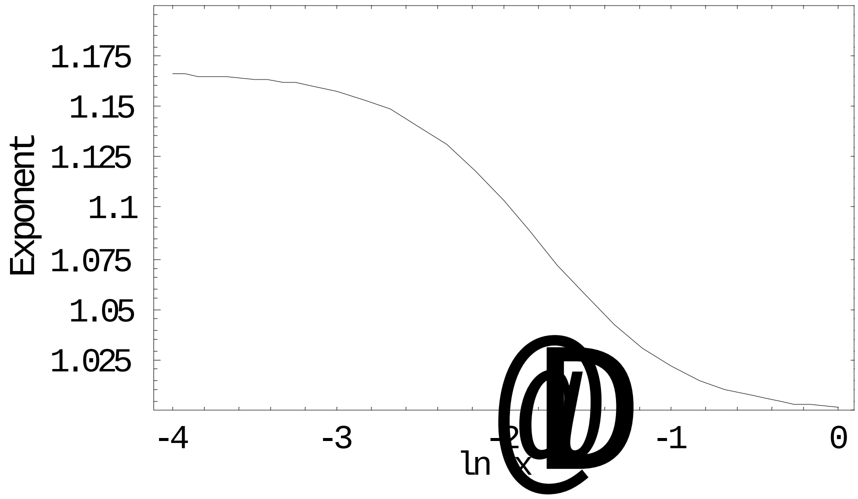

Fig. 3. The effective exponent of the susceptibility as a function of scaling variable $\ln[x]$, $x = g/\sqrt{r}$.

Also, the susceptibility amplitude ratio (χ_+/χ_-) is displayed in Fig. 4 using both models, the Gaussian model (solid curve) and the modified Gaussian model (dotted curve), as a function of $\mathrm{Log}[x]$, $x = g/\sqrt{r}$. As mentioned, the ratio in the Gaussian model starts at the pure Gaussian isotropic point, for example $x = 10^{-4}$, with the asymptotic value 2.83. At small values of x, $10^{-4} < x < 10^{-1}$, the ratio stays constant. For $x > 10^{-1}$, the ratio drops gradually in the crossover region to the mean field value of 2 at $x \gg 0$, where the pure uniaxial dipolar point is dominant. Regarding the modified Gaussian model (dotted curve) the ratio starts at the pure Gaussian isotropic point with the asymptotic value 5.1 and stays constant for the short interval, $10^{-4} < x < 10^{-3}$. The drop starts gradually and smoother than the drop in the Gaussian model, to the mean field value of 2 at $x \approx 10^{-1}$. It is worth noting that a small change in the scaling parameter from $x \approx 10^{-1}$ to $x \approx 10^{-2}$ doubled the rate's value.

Recently[4,5], accurate determination of the susceptibility amplitude ratio has been performed experimentally for polycrystalline samples of Ni_3Al and $Ni_{75}Al_{25}$. For Ni_3Al it was found that the ratio is $\frac{\chi_+}{\chi_-} = 4.0 \pm 0.1$, and the exponent is $\gamma = 1.075(20)$. Using Fig. 3, one can find that the experimental value will be located at $x \approx 10^{-2}$, which is close to, but not at, the critical uniaxial dipolar point. At $x \approx 10^{-2}$ one can find the effective exponent of the susceptibility as $\gamma = 1.1$. The same discussion could be implemented for the sample $Ni_{75}Al_{25}$.

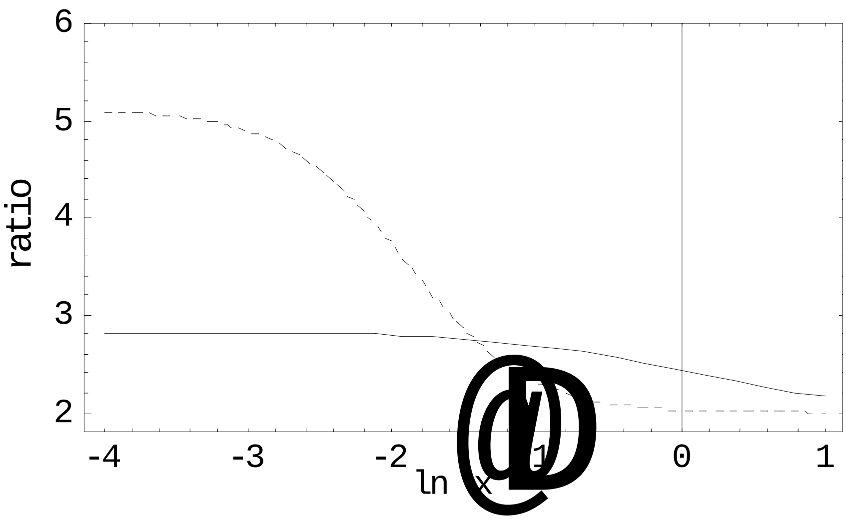

Fig. 4. The susceptibility amplitude ratio in the Gaussian model (solid curve) and the modified Gaussian model (dotted curve) as a function of scaling variable $\ln[x]$, $x = g/\sqrt{r}$.

From the above analysis, it is important to conclude the following points regarding the modified Gaussian model:

1- It increases the ratio at the pure Gaussian isotropic point from 2.8 (for the Gaussian model) to 5.2. This increment is mainly due to the effect of fluctuation, which is included partially in the modified model.
2- It makes the ratio decrease gradually and smoother over a fairly wide temperature range in the crossover region, than the decrease in the Gaussian model. Such gradual change has been seen experimentally[4,5].
3- It makes the pure uniaxial dipolar behavior start at a small value of x, $x \approx 10^{-2}$, which is physically reasonable. In the Gaussian model, the pure uniaxial dipolar behavior starts at $x \approx 1$.
4- It doubles the ratio, with a small change in the scaling variable, in the crossover region.

Conclusion

In summary, we have introduced here a modified model, the "modified Gaussian model", to calculate the non-universal susceptibility amplitude ratio for uniaxial dipolar ferromagnets. The crossover behavior between the uniaxial dipolar point and the usual point is discussed. Our calculations show that the modified Gaussian model improves our calculation over the Gaussian model. Also, the dipolar interaction is responsible for diminishing the susceptibility amplitude ratio in the crossover regions. This means that the dipolar interaction is crucial in calculating the ratio value close to the interesting point, especially when the renormalization-group theory is used. Generally, in the crossover region, the ratio remains dependent on all the parameters; however, the limiting values do not depend on the values of g or r. This independent property is to be expected in the asymptotic region. Last, but not least, we anticipate that our calculation will open the door to the experimentalist to predict the deviation of their measurements from the critical point, and also for the theoretician to carry out more precise calculations using full renormalization group theory to one and two-loop order for another system[18,19].

Acknowledgments

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APPENDIX A

To calculate the renormalized two-point vertex ($\hat{\Gamma}_R^{(2,0)}$), it is recommended to have the ε-expansion, to first order, for the used parameters. For example in the integral (see Equ. 7)

$$J(r,g) = \Gamma(\alpha)\left(\frac{g+\sqrt{r}}{2}\right)^{d-2} {}_2F_1(\alpha,\alpha;1;W) \tag{A.1}$$

by using $\varepsilon = 4-d$ and $\alpha = 1-d/2$ one can find $\alpha = \frac{\varepsilon}{2}-1$ and $d-2 = \varepsilon-2$. Consequently,

$$\Gamma(\alpha) = \Gamma\left(\frac{\varepsilon}{2}-1\right) = \frac{\Gamma\left(\frac{\varepsilon}{2}+1\right)}{\frac{\varepsilon}{2}\left(\frac{\varepsilon}{2}-1\right)} \approx -\frac{2}{\varepsilon}\Gamma\left(\frac{\varepsilon}{2}+1\right) \approx -\frac{2}{\varepsilon} \tag{A.2}$$

$${}_2F_1\left(\frac{\varepsilon}{2}-1,\frac{\varepsilon}{2}-1;1;W\right) = (1+W) - \varepsilon W + O(\varepsilon^2), \tag{A.3}$$

$$\begin{aligned}\left(\frac{g+\sqrt{r}}{2}\right)^{2-\varepsilon} &= \left(\frac{g+\sqrt{r}}{2}\right)^{2}\left(\frac{g+\sqrt{r}}{2}\right)^{-\varepsilon}, \\ &= \frac{1}{4}(g+\sqrt{r})^2(g+\sqrt{r})^{-\varepsilon}2^{\varepsilon}.\end{aligned} \tag{A.4}$$

Then

$$(g+\sqrt{r})^2 2^{-2+\varepsilon}[(1+W)-\varepsilon W] = \frac{1}{2}(g^2+r) + \frac{\varepsilon}{4}\left[2(g^2+r)\ln 2 - (g-\sqrt{r})^2\right] \tag{A.5}$$

$$\begin{aligned}J_1(r,g) &= \left(\frac{g+\sqrt{r}}{2}\right)^{2-\varepsilon} {}_2F_1\left(\frac{\varepsilon}{2}-1,\frac{\varepsilon}{2}-1;1;W\right) \\ &= -\frac{1}{2\varepsilon}(g+\sqrt{r})^{-\varepsilon}\left\{2(g^2+r)+\varepsilon\left[2(g^2+r)\ln 2-(g-\sqrt{r})^2\right]\right\}; \\ J_1(0,g) &= \\ &= -\frac{g^2}{2\varepsilon}(g+\sqrt{r})^{-\varepsilon}\left\{2+\varepsilon\left(\ln[4]-2\ln[g^2]+0.5\ln[g+\sqrt{r}]\right)\right\}\end{aligned} \tag{A.6}$$