Some Inequalities Concerning Polar Derivative of a Polynomial

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Abstract

Let \( p(z) = \sum_{v=0}^{n} a_v z^v \) be a polynomial of degree \( n \) having no zero in \( |z| < k \), \( k \leq 1 \), then Chanam et al. [Far East Journal of Mathematical Sciences, 127(1)(2020), 61-70] proved

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)|
- \frac{n|a_1|k^n}{1 + k^n} \left( \frac{1}{n} - \frac{k^n}{n - 2} \right) - |a_{n-1}|(1 - k^2), \quad \text{if } n > 2
\]

and

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)| - |a_{n-1}| \left( \frac{1 - k^n}{1 + k^n} \right), \quad \text{if } n = 2.
\]

provided \( |p'(z)| \) and \( |q'(z)| \) attain their maxima at the same point on the circle \( |z| = 1 \),
where

\[
q(z) = z^n p\left( \frac{1}{z} \right).
\]

In this paper, we extend the above inequalities to polar derivative of a polynomial. Further, we also prove an improved version of above inequalities into polar derivative.

Keywords and phrases: Inequalities, Polynomials, Zeros, Maximum modulus, Polar derivative of a polynomial.

AMS Subject Classification (2020): 15A18, 30C10, 30C15, 30A10.
1. INTRODUCTION

If \( p(z) \) is a polynomial of degree \( n \), then

\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}
\]

The above inequality is the well-known Bernstein inequality [3]. Inequality (1) is best possible and equality holds for the polynomial \( p(z) = \lambda z^n, \lambda \neq 0 \) being a complex number.

If we restrict to the class of polynomials having no zero in \( |z| < 1 \), then inequality (1) can be sharpened. In fact, Erdős conjectured and later Lax [11] proved that if \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < 1 \), then

\[
\max_{|z|=1} |z| = \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}
\]

Inequality (2) is sharp for polynomials having their zeros on \( |z| = 1 \).

The polar derivative of a polynomial \( p(z) \) of degree \( n \) with respect to a real or complex number \( \alpha \), denoted by \( D_\alpha p(z) \) is defined as

\[
D_\alpha p(z) = np(z) + (\alpha - z)p'(z).
\]

The polynomial \( D_\alpha p(z) \) is of degree at most \( n - 1 \) and it generalizes the ordinary derivative in the sense that

\[
\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).
\]

Aziz and Shah [2] extended inequality (1) to polar derivative and proved that if \( p(z) \) is a polynomial of degree \( n \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[
\max_{|z|=1} |D_\alpha p(z)| \leq n|\alpha| \max_{|z|=1} |p(z)|. \tag{3}
\]

Further, Aziz [1] extended inequality (2) to polar derivative and proved that if \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[
\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2} (|\alpha| + 1) \max_{|z|=1} |p(z)|. \tag{4}
\]

It was asked by R.P. Boas that if \( p(z) \) is a polynomial of degree \( n \) not vanishing in \( |z| < k, k > 0 \), then how large can

\[
\left\{ \frac{\max_{|z|=1} |p'(z)|}{\max_{|z|=1} |p(z)|} \right\} \text{ be ?}
\]
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A partial answer to this problem was given by Malik [12], who proved for the case \( k \geq 1 \) that

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)|. \tag{5}
\]

Equality in (5) holds for \( p(z) = (z + k)^n \).

For the class of polynomials not vanishing in \( |z| < k, \ k \leq 1 \), the precise estimate for maximum of \( |p'(z)| \) on \( |z| = 1 \), in general, does not seem to be easily obtainable.

For quite some time, it was believed that if \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < k, \ k \leq 1 \), then the inequality analogous to (5) should be

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)|, \tag{6}
\]

till E.B. Saff gave the example \( p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right) \) to counter this belief.

There are many extensions of inequality (5) (see, for example Bidkham and Dewan [4], Dewan and Mir [8] and Chan and Malik [5]).

However, for the class of polynomials not vanishing in \( |z| < k, \ k \leq 1 \), Govil [9] proved inequality (6) with extra condition.

**Theorem 1.1.** If \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < k, \ k \leq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)|, \tag{7}
\]

provided \( |p'(z)| \) and \( |q'(z)| \) attain their maxima at the same point on the circle \( |z| = 1 \), where

\[
q(z) = z^n p\left(\frac{1}{z}\right).
\]

Recently, Chana et al. [6] improved Theorem 1.1 by involving some of the co-efficients of the polynomial. In fact, they proved

**Theorem 1.2.** If \( p(z) = \sum_{v=0}^{n} a_n z^n \) is a polynomial of degree \( n \geq 2 \) having no zero in \( |z| < k, \ k \leq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)| - \frac{n|a_1| k}{1 + k^n} \left(\frac{1^{k-1}}{n} - \frac{1}{n - 2}\right) - \frac{n|a_{n-1}|(1 - k^2)}{1 + k^n}, \quad \text{if } n > 2 \tag{8}
\]

and

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)| - \frac{n|a_{n-1}|(1 - k^n)}{1 + k^n}, \quad \text{if } n = 2. \tag{9}
\]
provided \( |p'(z)| \) and \( |q'(z)| \) attain their maxima at the same point on the circle \( |z| = 1 \), where
\[
q(z) = z^n p\left(\frac{1}{z}\right).
\]

In this paper, we first prove the following result which extends Theorem 1.2 to polar derivative of \( p(z) \). In fact, we prove

**Theorem 1.3.** If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \geq 2 \) having no zero in \( |z| < k, k \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq \frac{1}{k} \),
\[
\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)|
- \frac{n|a_1| k^2 (k|\alpha| - 1)}{1 + k^n} \left\{ \frac{1 - k^n}{nk^2} - \frac{1 - k^{n-2}}{(n - 2)} \right\}
- (1 - k^2)|n\alpha_n + \alpha\alpha_{n-1}|, \text{ if } n > 2, \tag{10}
\]
and
\[
\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)|
- \frac{n|a_{n-1}| k^{n-2} (1 - k)^2 (k|\alpha| - 1)}{2(1 + k^n)}
- (1 - k)|n\alpha_n + \alpha\alpha_{n-1}|, \text{ if } n = 2, \tag{11}
\]

provided \( |D_\alpha p(z)| \) and \( |D_\alpha q(z)| \) attain their maxima at the same point on the circle \( |z| = 1 \), where
\[
q(z) = z^n p\left(\frac{1}{z}\right).
\]

**Remark 1.4.** From the hypotheses of Theorem 1.3, \( |D_\alpha p(z)| \) and \( |D_\alpha q(z)| \) attain their maxima at the same point on \( |z| = 1 \). Further, if they are divided by \( |\alpha| \) and considering limit as \( \alpha \to \infty \), then they become \( |p'(z)| \) and \( |q'(z)| \) which attain their maxima at the same point on \( |z| = 1 \). Hence, dividing both sides of inequalities (10) and (11) as well as the quantities \( |D_\alpha p(z)| \) and \( |D_\alpha q(z)| \) by \( |\alpha| \) and taking respectively limit as \( \alpha \to \infty \), we readily get inequalities (8) and (9) of Theorem 1.2 along with the agreement that \( |p'(z)| \) and \( |q'(z)| \) attain their maxima at the same point on the circle \( |z| = 1 \).

Next, under the same set of hypotheses, we prove a result which further improves the bounds of Theorem 1.3. More precisely, we obtain
Theorem 1.5. If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \geq 2 \) having no zero in \( |z| < k, \, k \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq \frac{1}{k} \),

\[
\max_{|z|=1} |D_{\alpha} p(z)| \leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)| - \frac{n(|\alpha| + k^n-1)}{1 + k^n} \min_{|z|=k} |p(z)| - n|a_1|k^2(k|\alpha| - 1) \left\{ \frac{1 - k^n}{nk^2} - \frac{1 - k^{n-2}}{(n-2)} \right\} - (1 - k^2)|n\overline{a}_n + \alpha\overline{a}_{n-1}|, \text{ if } n > 2,
\]

and

\[
\max_{|z|=1} |D_{\alpha} p(z)| \leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)| - \frac{n(|\alpha| + k^n-1)}{1 + k^n} \min_{|z|=k} |p(z)| - \frac{n|a_{n-1}|(1 - k^n)(k|\alpha| - 1)}{2kn^{2}(1 + k^n)} - (1 - k^2)|n\overline{a}_n + \alpha\overline{a}_{n-1}|, \text{ if } n = 2
\]

provided \( |D_{\alpha} p(z)| \) and \( |D_{\alpha} q(z)| \) attain their maxima at the same point on the circle \( |z| = 1 \), where

\[
q(z) = z^n p\left(\frac{1}{z}\right).
\]

Remark 1.6. If we adopt the similar argument of Remark 1.4 in Theorem 1.5, we get the following result proved by Chanam et al. [6, Theorem 1.3].

Theorem 1.7. If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \geq 2 \) having no zero in \( |z| < k, \, k \leq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\} - \frac{n|a_1|k^{\frac{n-2}{n}}}{1 + k^n} \left( 1 - \frac{1}{n^2} \right) - |a_{n-1}|(1 - k^2), \text{ if } n > 2
\]

and

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\} - |a_{n-1}|(1 + k^n), \text{ if } n = 2
\]

provided \( |p'(z)| \) and \( |q'(z)| \) attain their maxima at the same point on the circle \( |z| = 1 \), where

\[
q(z) = z^n p\left(\frac{1}{z}\right).
\]

2. LEMMAS.

For the proofs of the theorems, we will use the following lemmas. The first lemma is a special case of a result due to Govil and Rahman [10].
Lemma 2.1. If $p(z)$ is a polynomial of degree $n$, then on $|z| = 1$,

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|,$$  \hfill (15)

where

$$q(z) = z^n p\left(\frac{1}{z}\right).$$

Lemma 2.2. If $p(z)$ is a polynomial of degree $n$ and $\alpha$ is any real or complex number, then on $|z| = 1$,

$$|D_\alpha p(z)| + |D_\alpha q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)|,$$  \hfill (16)

where

$$q(z) = z^n p\left(\frac{1}{z}\right).$$

Lemma 2.2 was proved by Aziz [1, Lemma 2] in more general form. However, we present a simple proof of this lemma which we think is new, simply by using definition of polar derivative of a polynomial and Lemma 2.1 due to Govil and Rahman [10].

Proof of Lemma 2.2. Let $q(z) = z^n p\left(\frac{1}{z}\right)$. Then it is easy to verify that on $|z| = 1$,

$$|q'(z)| = |np(z) - zp'(z)|. \hfill (17)$$

Now, for every real or complex number $\alpha$, the polar derivative of $p(z)$ with respect to $\alpha$ is

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \hfill (18)$$

This implies on $|z| = 1$,

$$|D_\alpha p(z)| \leq |np(z) - zp'(z)| + |\alpha||p'(z)|. \hfill (19)$$

Using (17) in (19), we have on $|z| = 1$,

$$|D_\alpha p(z)| \leq |q'(z)| + |\alpha||p'(z)|. \hfill (20)$$

Similarly, on $|z| = 1$,

$$|D_\alpha q(z)| \leq |p'(z)| + |\alpha||q'(z)|. \hfill (21)$$

Adding (20) and (21), we have

$$|D_\alpha p(z)| + |D_\alpha q(z)| \leq (|\alpha| + 1) \{ |p'(z)| + |q'(z)| \}. \hfill (22)$$
Using Lemma 2.1 in (22), we get
\[ |D_{\alpha}p(z)| + |D_{\alpha}q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)|, \] (23)
which completes the proof of Lemma 2.2.

The next lemma is due to Mir [13].

**Lemma 2.3.** If \( p(z) = \sum_{v=0}^{n} a_v z^n \) is a polynomial of degree \( n \geq 2 \) having all its zeros in \( |z| \leq k, k \geq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq k \),
\[
\max_{|z|=1} |D_{\alpha}p(z)| \geq n \left( \frac{|\alpha| - k}{1 + k^n} \right) \left\{ \max_{|z|=1} |p(z)| + \frac{|a_{n-1}|}{k} \left( \frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) \right\}
+ \left( 1 - \frac{1}{k^2} \right) |na_0 + \alpha a_1|, \text{ if } n > 2
\] (24)
and
\[
\max_{|z|=1} |D_{\alpha}p(z)| \geq n \left( \frac{|\alpha| - k}{1 + k^n} \right) \left\{ \max_{|z|=1} |p(z)| + \frac{|a_{n-1}|}{2k} (k - 1)^2 \right\}
+ \left( 1 - \frac{1}{k} \right) |na_0 + \alpha a_1|, \text{ if } n = 2.
\] (25)

**Lemma 2.4.** If \( p(z) = \sum_{v=0}^{n} a_v z^n \) is a polynomial of degree \( n \geq 2 \) having all its zeros in \( |z| \leq k, k \geq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq k \),
\[
\max_{|z|=1} |D_{\alpha}p(z)| \geq n \left( \frac{|\alpha| - k}{1 + k^n} \right) \left\{ \max_{|z|=1} |p(z)| + \frac{|a_{n-1}|}{k} \left( \frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) \right\}
+ \frac{k^{n-1}|\alpha| + 1}{k^{n-1}|\alpha| - k^n} \min_{|z|=k} |p(z)|
+ \left( 1 - \frac{1}{k^2} \right) |na_0 + \alpha a_1|, \text{ if } n > 2
\] (26)
and
\[
\max_{|z|=1} |D_{\alpha}p(z)| \geq n \left( \frac{|\alpha| - k}{1 + k^n} \right) \left\{ \max_{|z|=1} |p(z)| + \frac{|a_{n-1}|}{k} k^{n-3}(k - 1)^n \right\}
+ \frac{k^{n-1}|\alpha| + 1}{k^{n-1}|\alpha| - k^n} \min_{|z|=k} |p(z)|
+ \left( 1 - \frac{1}{k} \right) |na_0 + \alpha a_1|, \text{ if } n = 2
\] (27)
This result was proved by Dewan and Chanam [7].

3. PROOFS OF THEOREMS.

We first prove Theorem 1.5.

**Proof of Theorem 1.5.** Let \( p(z) = \sum_{v=0}^{n} a_v z^n \) be a polynomial of degree \( n \geq 2 \) having no zero in \( |z| < k, k \leq 1 \). In other words, \( p(z) \) has all its zeros in \( |z| \geq k, k \leq 1 \) and
hence all the zeros of \( q(z) = z^n p\left(\frac{1}{z}\right) \) lies in \(|z| \leq 1/k, 1/k \geq 1\).

Applying Lemma 2.4 on \( q(z) \), for \(|\alpha| \geq \frac{1}{k} \), we have

\[
\max_{|z|=1} |D_\alpha q(z)| \geq n \left( \frac{|\alpha| - \frac{1}{k}}{1 + \frac{1}{k}} \right) \left\{ \max_{|z|=1} |q(z)| + \frac{|a_1|}{k} \left( \frac{\frac{1}{k} - 1 - \frac{1}{n} - 2}{n} \right) \right. \\
+ \left. \frac{1}{k^{n-1}} |\alpha| + 1 \min_{|z|=k} |q(z)| \right\} + \left( 1 - \frac{1}{k^2} \right) |n\alpha_n + \alpha\alpha_{n-1}|, \quad \text{if } n > 28
\]

and

\[
\max_{|z|=1} |D_\alpha q(z)| \geq n \left( \frac{|\alpha| - \frac{1}{k}}{1 + \frac{1}{k}} \right) \left\{ \max_{|z|=1} |q(z)| + \frac{|a_1|}{k^{n-1}} \left( \frac{1 - k^n}{2} \right) \right. \\
+ \left. \frac{1}{k} |\alpha| + 1 \min_{|z|=k} |q(z)| \right\} + \left( 1 - k^2 \right) |n\alpha_n + \alpha\alpha_{n-1}|, \quad \text{if } n > 29
\]

Which is equivalent to

\[
\max_{|z|=1} |D_\alpha q(z)| \geq \frac{nk^{n-1} (k|\alpha| - 1)}{1 + k^n} \\
\times \left[ \max_{|z|=1} |q(z)| + |a_1| k\left\{ \frac{1 - k^n}{nk^n} - \frac{1 - k^{n-2}}{(n-2)k^{n-2}} \right\} \\
+ \frac{k(|\alpha| + k^{n-1})}{k|\alpha| - 1} \min_{|z|=k} |q(z)| \right] + (1 - k^2) |n\alpha_n + \alpha\alpha_{n-1}|, \quad \text{if } n > 30
\]

and

\[
\max_{|z|=1} |D_\alpha q(z)| \geq \frac{nk^{n-1} (k|\alpha| - 1)}{1 + k^n} \left\{ \max_{|z|=1} |q(z)| + \frac{|a_1| (1 - k^n)}{2k^{n-3}} \right. \\
+ \left. \frac{k(|\alpha| + k^{n-1})}{k|\alpha| - 1} \min_{|z|=k} |q(z)| \right\} + (1 - k) |n\alpha_n + \alpha\alpha_{n-1}|, \quad \text{if } n > 31
\]

Now,

\[
\min_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \min_{|z|=k} |p(z)|. \tag{32}
\]

Using (32) in (30) and (31), we have

\[
\max_{|z|=1} |D_\alpha q(z)| \geq \frac{nk^{n-1} (k|\alpha| - 1)}{1 + k^n} \\
\times \left[ \max_{|z|=1} |q(z)| + |a_1| k\left\{ \frac{1 - k^n}{nk^n} - \frac{1 - k^{n-2}}{(n-2)k^{n-2}} \right\} \\
+ \frac{k(|\alpha| + k^{n-1})}{k^n(k|\alpha| - 1)} \min_{|z|=k} |p(z)| \right] + (1 - k^2) |n\alpha_n + \alpha\alpha_{n-1}|, \quad \text{if } n > 33
\]
and
\[
\max_{|z|=1} |D_\alpha q(z)| \geq \frac{n k^{n-1}(k|\alpha| - 1)}{1 + k^n} \left\{ \max_{|z|=1} |q(z)| + |a_1| \left( \frac{1 - k^n}{2 k^{2n-3}} \right) \right. \\
+ \left. \frac{k(|\alpha| + k^{n-1})}{k^{n-1}(k|\alpha| - 1)} \min_{|z|=k} |p(z)| \right\} + (1 - k)|n\alpha_n + \alpha a_{n-1}|, \quad \text{if } n = (24)
\]

Also, since on \(|z| = 1|, \, |p(z)| = |q(z)|,\) inequalities (33) and (34) can be written as
\[
\max_{|z|=1} |D_\alpha q(z)| \geq \frac{n k^{n-1}(k|\alpha| - 1)}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| + |a_1| \left( \frac{1 - k^n}{n k^n} - \frac{1 - k^{n-2}}{(n - 2)k^{n-2}} \right) \right. \\
+ \left. \frac{(|\alpha| + k^{n-1})}{k^{n-1}(k|\alpha| - 1)} \min_{|z|=k} |p(z)| \right\} + (1 - k^2)|n\alpha_n + \alpha a_{n-1}|, \quad \text{n > (25)}
\]

and
\[
\max_{|z|=1} |D_\alpha q(z)| \geq \frac{n k^{n-1}(k|\alpha| - 1)}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| + |a_1| \left( \frac{1 - k^n}{2 k^{2n-3}} \right) \right. \\
+ \left. \frac{(|\alpha| + k^{n-1})}{k^{n-1}(k|\alpha| - 1)} \min_{|z|=k} |p(z)| \right\} + (1 - k)|n\alpha_n + \alpha a_{n-1}|, \quad \text{if } n = (26)
\]

By Lemma 2.2, on \(|z| = 1|,
\[
|D_\alpha p(z)| + |D_\alpha q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)|.
\]

(37)

Let \(z_0\) be a point on \(|z| = 1\) such that \(\max_{|z|=1} |D_\alpha q(z)| = |D_\alpha q(z_0)|.\) Since \(|D_\alpha p(z)|\) and \(|D_\alpha q(z)|\) attain their maxima at the same point on \(|z| = 1\) with \(|\alpha| \geq \frac{1}{k}\), we have
\[
\max_{|z|=1} |D_\alpha p(z)| = |D_\alpha p(z_0)|.
\]

Thus, in particular (37) gives
\[
\max_{|z|=1} |D_\alpha q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)|.
\]

(38)

Combining (38) with (35) and (36), we have
\[
n(|\alpha| + 1) \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)| \geq \frac{n k^{n-1}(k|\alpha| - 1)}{1 + k^n} \\
\times \left\{ \max_{|z|=1} |p(z)| + |a_1| \left( \frac{1 - k^n}{n k^n} - \frac{1 - k^{n-2}}{(n - 2)k^{n-2}} \right) \right. \\
+ \left. \frac{(|\alpha| + k^{n-1})}{k^{n-1}(k|\alpha| - 1)} \min_{|z|=k} |p(z)| \right\} \\
+ \left( 1 - k^2 \right)|n\alpha_n + \alpha a_{n-1}|, \quad \text{if } n > 2
\]

(39)
and
\[ n(|\alpha| + 1) \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)| \geq \frac{n k^{n-1} |k|\alpha - 1}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| + |a_1| \right\} \frac{(1 - k)^n}{2 k^{2n-3}} \\
+ \frac{|\alpha| + k^{n-1}}{k^{n-1} |k|\alpha - 1} \min_{|z|=k} |p(z)| \right\} \\
+ (1 - k) n \alpha \overline{\alpha} + \alpha \overline{\alpha}_{n-1}, \text{ if } n = 2, \quad (40) \]

which is equivalent to
\[
\max_{|z|=1} |D_\alpha p(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)| - \left\{ \frac{n k^{n-1} |k|\alpha - 1}{1 + k^n} \right\} \max_{|z|=1} |p(z)| \\
- \left\{ \frac{n|a_1| k^{n-1} |k|\alpha - 1}{1 + k^n} \right\} \left\{ \frac{1 - k^n}{n k^n} - \frac{1 - k^{n-2}}{(n - 2)} \right\} \\
- \left\{ \frac{n k^{n-1} |k|\alpha - 1}{1 + k^n} \right\} \frac{|\alpha| + k^{n-1}}{k^{n-1} |k|\alpha - 1} \min_{|z|=k} |p(z)| \\
- (1 - k^2) n \alpha \overline{\alpha} + \alpha \overline{\alpha}_{n-1}, \text{ if } n > 2, \quad (41) \]

and
\[
\max_{|z|=1} |D_\alpha p(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)| - \left\{ \frac{n k^{n-1} |k|\alpha - 1}{1 + k^n} \right\} \max_{|z|=1} |p(z)| \\
- \left\{ \frac{n|a_1| k^{n-1} (|k|\alpha - 1)}{1 + k^n} \right\} \left\{ \frac{(1 - k^n)}{2 k^{2n-3}} \right\} \\
- \left\{ \frac{n k^{n-1} (|k|\alpha - 1)}{1 + k^n} \right\} \frac{|\alpha| + k^{n-1}}{k^{n-1} (|k|\alpha - 1)} \min_{|z|=k} |p(z)| \\
- (1 - k) n \alpha \overline{\alpha} + \alpha \overline{\alpha}_{n-1}, \text{ if } n = 2, \quad (42) \]

which on simplification gives
\[
\max_{|z|=1} |D_\alpha p(z)| \leq n(|\alpha| + k^n + k^{n-1} + 1) \max_{|z|=1} |p(z)| \\
- \frac{n|a_1| k^2 (|k|\alpha - 1)}{1 + k^n} \left\{ \frac{1 - k^n}{n k^2} - \frac{1 - k^{n-2}}{(n - 2)} \right\} \\
- \frac{n(|\alpha| + k^{n-1})}{1 + k^n} \min_{|z|=k} |p(z)| \\
- (1 - k^2) |n \alpha \overline{\alpha} + \alpha \overline{\alpha}_{n-1}|, \text{ if } n > 2, \quad (43) \]

and
\[
\max_{|z|=1} |D_\alpha p(z)| \leq n(|\alpha| + k^n + k^{n-1} + 1) \max_{|z|=1} |p(z)| - \frac{n|a_1| (|k|\alpha - 1)(1 - k^n)}{2 k^{n-2}(1 + k^n)} \\
- \frac{n(|\alpha| + k^{n-1})}{1 + k^n} \min_{|z|=k} |p(z)| - (1 - k) |n \alpha \overline{\alpha} + \alpha \overline{\alpha}_{n-1}|, \text{ if } n = 2 \quad (44) \]
which is the proof of Theorem 1.5.

**Proof of Theorem 1.3** The proof of this theorem follows on the same lines as that of Theorem 1.5 but instead of applying Lemma 2.4, we apply Lemma 2.3 and we omit it.

**REFERENCES**


