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Hyers-Ulam Stability of Certain Class of Nonlinear Second Order Differential Equations

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Abstract

We investigate the Hyers-Ulam stability of certain classes of nonlinear second order differential equations using a nonlinear generalisation of Gronwall-Bellman integral inequality known as Bihari integral inequality.

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Keywords: Hyers-Ulam stability, Bihari integral inequality, Nonlinear second order differential equations.

1. Introduction

In 1940, Ulam [30] gave a wide-rage talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among these was the question concerning the stability of homomorphisms. Hyers [10] solved the problem of Ulam for additive functions defined on Banach spaces thus: If X and Y are real Banach spaces and $\epsilon > 0$. Then for every function $f : X \to Y$ satisfying $||f(x+y) - f(x) - f(y)|| \leq \epsilon$, for all $x, y, \in X$, there exists a unique additive function $A : X \to Y$ with the property $||f(x) - A(x)|| \leq \epsilon$ for all $x \in X$. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians [8, 9, 11, 16, 22].

Obloza [20, 21] investigated the Hyers-Ulam stability of linear differential equation and connections between Hyers and Lyapunov stability of the ordinary differential equation. Alsina and Ger [1] continued and investigated the Hyers-Ulam stability of the differential $g' = g$. They proved that if a differentiable function $y : \mathbf{I} \to \mathbf{R}$ satisfies $|y' - y| \leq \epsilon$ for all $t \in \mathbf{I}$, then there exists a differentiable function $g : \mathbf{I} \to \mathbf{R}$ satisfying $g'(t) = g(t)$ for any $t \in \mathbf{I}$ such that $|g(t) - y(t)| \leq 3\epsilon$ for all $t \in \mathbf{I}$. The result by Alsina and Ger has been generalised by others including: Miura [17, 18], Takhasi [29, 30] and Jung [11, 12, 13, 14, 15] who proved the Hyers-Ulam stability of linear differential equations. Rus [25, 26] investigated the Hyers-Ulam stability of differential and integral equations using the Gronwall lemma [7] and the technique of weakly Picard operators. Recently, Qusuay [24] applied the Grownwall lemma to investigate the Hyers-Ulam stability of the form $u''(t) + f(t, u(t)) = 0$. and Emden-Fowler nonlinear differential equation of second order $u''(t) + h(t)u(t)^{\alpha} = 0$ for the case where α is a positive odd integer. Quasuay did not consider the case when α is even integers and the function f is of the form $f(t, u(t), u'(t))$. These are the problems we consider in this paper using nonlinear generalisation of Gronwall-Bellman [2, 3] called Bihari inequality [4, 5].

The result obtained in this paper generalises the works of Quasuay [23] and Qarawani [24] on nonlinear second order differential equations. In this paper, we focus on the investigation of the Hyers-Ulam stability of the nonlinear second order differential equations given below.

$$u''(t) + f(t, u(t)) = 0 \tag{1.1}$$

$$u''(t) + f(t, u(t), u'(t)) = 0 \tag{1.2}$$

2. Preliminaries

In this section, we shall state the Bihari lemma and other useful results and definitions.

Lemma 2.1. [4, 5] Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \geq 0$, then the inequality

$$u(t) \leq K + M \int_a^t f(s)\omega(u(s))ds, \quad a \leq t < b \tag{2.1}$$

implies the inequality

$$u(t) \leq \Omega^{-1}\left(\Omega(k) + M \int_a^t f(s)ds\right), \quad a \leq t \leq b' \leq b \tag{2.2}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u \tag{2.3}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[a, b']$ of $[a, b]$ such that

$$\Omega(k) + M \int_a^t f(s)ds \in Dom(\Omega^{-1})$$

Theorem 2.2. (Generalised First Mean Value Theorem) [19, 27] If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbf{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that

$$\int_{t_0}^t g(s)f(s)ds = g(\xi) \int_{t_0}^t f(s)ds$$

Definition 2.3. [6] A function ω is said to belong to a class H if it satisfies the following conditions

i $\omega(u) > 0$ is nondecreasing and $\omega \in \mathbf{C}^0$ for $u > 0$

ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$ when ω is a positive, nondecreasing function defined and continuous on $\mathbf{I}$.

Definition 2.4. The equation (1.1) with the initial condition $u(t_0) = u'(t_0) = 0$ has Hyers-Ulam stability if there exists a positive constant $K > 0$ with following property. For every $\epsilon > 0$ $u \in C^2(\mathbf{I})$, if

$$|u''(t) + f(t, u(t))| \leq \epsilon \tag{2.1}$$

then, there exist a solution $u_0(t) \in C^2(\mathbf{I})$ of the equation (1.1), such that

$$|u(t) - u_0(t)| \leq K\epsilon.$$

Definition 2.5. Equation (1.2) with the initial condition $u(t_0) = u'(t_0) = 0$ has Hyers-Ulam stability if there exists a positive constant $K > 0$ with following property. For every $\epsilon > 0$ $u \in C^2(\mathbf{I})$, if

$$|u''(t) + f(t, u(t), u'(t))| \leq \epsilon \tag{2.2}$$

then, there exist a solution $u_0(t) \in C^2(\mathbf{I})$ of the equation (1.2), such that

$$|u(t) - u_0(t)| \leq K\epsilon.$$

3. Main Result

As earlier stated, we first investigate the Hyers-Ulam stability of the second order nonlinear differential equation of the form (1.1) where the function $f(t, u(t))$ satisfies the condition

$$|f(t, u(t))| \leq \phi(t)\omega(|u(t)|) \tag{3.1}$$

Where $\phi(t)$ is a continuous, nonnegative function for $t \geq t_0$ and $\omega(u)$ is a continuous, nondecreasing, nonnegative function for $u > 0$. Besides, the function $f(t, u(t))$ is continuous on $D := \{t, u : t \in [t_0, \infty), u \in \mathbf{I}\}$.

Theorem 3.1. Let $\int_{t_0}^{\infty} |u'(s)|ds \leq L$ for $L > 0$ be assumed and let the function $f(t, u(t))$ satisfies the following conditions:

c_1 $u(t) \leq f(t, u(t))u(t)$, where $f(t, u(t)) > 1$ for all $t \geq t_0$.

c_2 $\dfrac{f'(t, u(t))u(t)}{f(t, u(t))} = g(t, u(t))$ g a positive, continuous function.

c_3 $|f(t, u(t))| \leq \phi(t)\omega(|u(t)|)$, where $\omega(t)$ belongs to the class H

c_4 $\Omega(r) = \displaystyle\int_{r_0}^{r} \frac{ds}{\omega(s)}$ $r_0 \geq 0, r \geq r_0$

If $u : \mathbf{I} \to \mathbf{I}$ satisfying $u \in C^2(\mathbf{I})$ and the inequality

$$|u''(t) + f(t, u(t))| \leq \epsilon \text{ for all } t \geq t_0 \text{ and for some } \epsilon > 0,$$

then there exist a solution $u_0(t) \in C^2(\mathbf{I})$ of the differential equation (1.1) such that $|u(t) - u_0(t)| \leq K\epsilon$ for any $t \geq 0$, provided $\int_{t_0}^{\infty} \phi(s) < M < \infty$ and K = $L\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))|M\right)$. Therefore, equation (1.1) has Hyers-Ulam stability with initial condition $u(t_0) = u'(t_0) = 0$.

Proof. Multiplying (2.1) by $|u'(t)|$ to get

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t, u(t))u'(t) \leq \epsilon|u'(t)| \tag{3.2}$$

for all $t \geq t_0$. Integrating each term from t_0 to t, then,

$$-\epsilon \int_{t_0}^{t} |u'(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s, u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} |u'(s)|ds$$

for any $t \geq t_0$, Integrating by part, let $\int_{t_0}^{\infty} |u'(s)|ds \leq L$ for $L > 0$ and $f_u(t, u(t)) \leq 0$.

$$-\epsilon L \leq \frac{1}{2}u'(t)^2 + f(t, u(t))u(t) - \int_{t_0}^{t} f'(s, u(s))u(s)ds \leq \epsilon L$$

for all $t \geq t_0$. Then, it follows that

$$f(t, u(t))u(t) \leq \epsilon L + \int_{t_0}^{t} f'(s, u(s))u(s)ds \tag{3.3}$$

Applying c_1 to (3.3),

$$u(t) \leq \epsilon L + \int_{t_0}^{t} f'(s, u(s))u(s)ds \quad \text{for} \quad t \geq t_0 \tag{3.4}$$

We write (3.4) as,

$$u(t) \leq \epsilon L + \int_{t_0}^{t} \frac{f'(s, u(s))u(s)}{f(s, u(s))} f(s, u(s))ds \quad \text{for} \quad t \geq t_0 \tag{3.5}$$

Applying c_2 and using generalised Mean value theorem in a closed region D.

$$u(t) \leq \epsilon L + g(\xi, u(\xi)) \int_{t_0}^{t} f(s, u(s))ds \quad \text{for} \quad t \leq t_0$$

$$\leq \epsilon L + |g(\xi, u(\xi))| \int_{t_0}^{t} |f(s, u(s))|ds \quad \text{for} \quad t \leq t_0$$

$$\leq \epsilon L + |g(\xi, u(\xi))| \int_{t_0}^{t} |f(s, u(s))|ds \quad \text{for} \quad t \leq t_0$$

Since $\epsilon L > 0$, we have

$$\frac{|u(t)|}{\epsilon L} \leq 1 + |g((\xi), u(\xi))| \int_{t_0}^{t} \phi(s)\omega\left(\frac{|u(s)|}{\epsilon L}\right) ds \quad t \leq t_0 \tag{3.6}$$

Setting $v(t) =$ R .H.S (3.6)
since ω is nondecreasing we have

$$\begin{aligned} 0 < \omega\left(\frac{|u(t)|}{\epsilon L}\right) &\leq \omega(v(t)) \\ v'(t) &= |g((\xi), u(\xi))|\phi(t)\omega\left(\frac{|u(t)|}{\epsilon L}\right) \\ &\leq |g(\xi), u(\xi))|\phi(t)\omega(v(t)) \end{aligned}$$

then,

$$\frac{v'(t)}{\omega(v(t))} \leq |g(\xi.u(\xi))|\phi(t)$$

By the definition of Ω, this gives

$$\frac{d\Omega(v(t))}{dt} \leq |g(\xi.u(\xi))|\phi(t)$$

Integrating from t_0 to t gives

$$\Omega(v(t)) - \Omega(v(t_0)) \leq |g(\xi.u(\xi))| \int_{t_0}^{t} \phi(s)ds$$

since $v(t_0) = 1$ and $\Omega^{-1}(u)$ being increasing. Also we have

$$v(t) \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))| \int_{t_0}^{t} \phi(s)ds\right)$$

and finally from (3.6) we obtain

$$\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))| \int_{t_0}^{t} \phi(s)ds\right), \quad t \geq t_0$$

As $t \to \infty$ then,

$$\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))|M\right)$$

provided $\lim\limits_{t\to\infty} \int_{t_0}^{t} \phi(s)ds \leq M < \infty$

Hence,

$$|u(t)| \leq \epsilon L\left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))|M\right)\right) \quad \text{for all} \quad t \geq t_0.$$

$$|u(t) - u_0(t)| \leq K\epsilon$$

Where $K = \epsilon L\left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))|M\right)\right)$. ■

Hence, the equation (1.1) has Hyers-Ulam stability.

Example 3.2. Consider Hyers-Ulam stability of the nonlinear differential equation of the form

$$u''(t) + t^{-4}u^2 \exp(u(t)) = 0. \tag{3.7}$$

taking

$$f(t, u(t)) = t^{-4}u^2(t) \exp(u(t))$$

and allow

$$\omega(u) = u^2 \exp(u(t)), \quad \phi(t) = t^{-4}$$

Where $u(t_0) = u'(t_0) = 0$ and $u_0(t) = 0$.

Therefore, the equation (3.7) is Hyers-Ulam stable.

In our next result we consider the Hyers-Ulam stability of the nonlinear differential equation (1.2), Where the function $f(t, u(t), u'(t))$ is continuous on $D = \{t, u, u' : t \in [t_0, \infty), u, u' \in \mathbf{I}\}$ and satisfies some conditions to be prescribed later.

Theorem 3.3. Let $\int_{t_0}^{\infty} |u'(s)|ds \leq L$ for $L > 0$ be assumed and let function $f(t, u(t), u'(t))$ satisfies the following conditions:

H_1 $u(t) \leq f(t, u(t), u'(t))u(t)$, where $f(t, u(t), u'(t)) > 1$ for all $t \geq t_0$.

H_2 $\dfrac{f'(t, u(t), u'(t))u(t)}{f(t, u(t), u(t))} = g(t, u(t), u'(t))$, g a positive, continuous function.

H_3 $|f(t,u(t),u(t))| \leq h(t)\omega(|u(t)|)|u'(t)|$ where $\omega(t)$ belongs to the class H, for $s > 0$ the function $\omega(s)$ is nondecreasing. Where $h, \quad \omega : \mathbf{I} \to \mathbf{I}$ are nonnegative, continuous functions

H_4 $\Omega(r) = \int_{r_0}^{r} \frac{ds}{s\omega(s)} \quad r_0 \geq 0, r \geq r_0$

If $u \in C^2(\mathbf{I})$, $|u'(t)| \leq \frac{|u(t)|}{\epsilon L}$ and the inequality

$$|u''(t) + f(t,u(t),u'(t))| \leq \epsilon \text{ for all } \quad t \geq t_0 \text{ and for some } \epsilon > 0,$$

then there exist a solution $u_0(t) \in C^2(\mathbf{I})$ of the differential equation (1.2) such that $|u(t) - u_0(t)| \leq K\epsilon$ for any $t \geq 0$, provided
$\int_{t_0}^{\infty} h(s)ds < M < \infty$ and $\Omega^{-1}\left(\Omega(1) + |g(\xi,u(\xi),u'(\xi))|M\right) < \infty$, where $K = L\Omega^{-1}\left(\Omega(1) + |g(\xi,u(\xi),u'(\xi))|M\right)$.
Therefore, equation(1.2) has Hyers-Ulam stability with initial condition $u(t_0) = u'(t_0) = 0$.

Proof. Multiplying (2.2) by $|u'(t)|$ to get

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t,u(t),u'(t))u'(t) \leq \epsilon|u'(t)| \tag{3.8}$$

for all $t \geq t_0$. Integrating each term from t_0 to t, then,

$$-\epsilon \int_{t_0}^{t} |u(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} |u'(s)|ds$$

for any $t \geq t_0$.
Integrating by part, let $f_u(t,u(t),u'(t)) + f_{u'}(t,u(t),u'(t)) \leq 0$ and using hypothesis in the theorem, we have

$$-\epsilon L \leq \frac{1}{2}u'(t)^2 + f(t,u(t),u'(t))u(t) - \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds \leq \epsilon L$$

for all $t \geq t_0$. Then,

$$f(t,u(t),u'(t))u(t) \leq \epsilon L + \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds$$

Using H_1

$$u(t) \leq \epsilon L + \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds \quad \text{for} \quad t \geq t_0 \tag{3.9}$$

We can write (3.9) as

$$u(t) \leq \epsilon L + \int_{t_0}^{t} \frac{f'(s,u(s),u'(s))u(s)}{f(s,u(s),u'(s))} f(s,u(s),u'(s))ds \quad \text{for} \quad t \geq t_0$$

Applying H_2

$$u(t) \leq \epsilon L + \int_{t_0}^{t} g(s,u(s),u'(s))f(s,u(s),u'(s))ds \quad \text{for} \quad t \geq t_0$$

By application of generalised Mean value theorem of integral in the closed region D,

$$u(t) \leq \epsilon L + g(\xi,u(\xi),u'(\xi)) \int_{t_0}^{t} f(s,u(s),u'(s)ds \quad \text{for} \quad t \geq t_0$$

$$|u(t)| \leq \epsilon L + |g(\xi,u(\xi),u'(\xi))| \int_{t_0}^{t} |f(s,u(s),u'(s))|ds \quad \text{for} \quad t \geq t_0$$

$$|u(t)| \leq \epsilon L + |g(\xi,u(\xi),u'(\xi))| \int_{t_0}^{t} h(s)\omega(|u(s)|)|u'(s)|ds \quad by \quad H_3$$

$$\frac{|u(t)|}{\epsilon L} \leq 1 + |g(\xi,u(\xi),u'(\xi))| \int_{t_0}^{t} h(s)\omega(\frac{|u(s)|}{\epsilon L})|u'(s)|ds \tag{3.10}$$

Setting $z(t) =$ R.H.S of (3.10)
Hence,

$$z(t) \leq 1 + |g(\xi,u(\xi),u'(\xi)| \int_{t_0}^{t} h(s)\omega(z(s))z(s)ds \tag{3.11}$$

Setting $v(t)$= R.H.S of (3.11) since ω is nondecreasing we have

$$\begin{aligned} 0 < \omega(z(t)) &\leq \omega(v(t)) \\ v'(t) &= |g(\xi,u(\xi),u'(\xi))|h(t)\omega(z(t))z(t) \\ &\leq |g(\xi,u(\xi),u'(\xi))|h(t)\omega(v(t))v(t) \end{aligned}$$

$$\frac{v'(t)}{\omega(v(t))v(t)} \leq |g(\xi,u(\xi),u'(\xi))|h(t)$$

Application of H_4, this gives,

$$\frac{d\Omega(v(t))}{dt} \leq |g((\xi),u(\xi),u'(\xi))|h(t)$$

Integrating from t_0 to t gives

$$\Omega(v(t)) - \Omega(v(t_0)) \leq |g((\xi),u(\xi),u'(\xi))| \int_{t_0}^{t} h(s)ds$$

since $v(t_0) = 1$ and $\Omega^{-1}(u)$ being increasing also we have

$$z(t) \leq v(t) \leq \Omega^{-1}\left(\Omega(1) + |g((\xi),u(\xi),u'(\xi))| \int_{t_0}^{t} h(s)ds\right)$$

Finally from (3.11) we obtain

$$\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} h(s)ds\right) \quad \text{for} \quad t \geq t_0$$

As $t \to \infty$, then,

$$\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))|M\right)$$

provided $\lim_{t_0} \to \infty \int_{t_0}^{t} h(s)ds \leq M < \infty$
Hence,

$$|u(t)| \leq \epsilon L\left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))|M\right)\right) \quad \text{for all} \quad t \geq t_0.$$

Where,

$$K = L\left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi)u'(\xi)|M\right)\right) \quad \text{for all} \quad t \geq t_0.$$

Hence, it holds that $|u(t)| \leq K\epsilon$ for any $t \geq t_0$, with initial condition $u_0(t) = u'(t) = 0$ satisfies (1.2) and $u_0 \in C^2(\mathbf{I})$ such that $|u(t) - u_0(t) \leq K\epsilon$. ■

Example 3.4. To investigate Hyers-Ulam stability of the second order nonlinear differential equation of the form

$$u''(t) + (2t)^{-4}u^2 \exp(u'(t))u'(t) = 0 \qquad (3.12).$$

We take

$$f(t, u(t), u'(t)) = (2t)^{-4}u^2 \exp(u'(t))u'(t)$$

and allow $\omega(u) = u^2$, $h(t) = (2t)^{-4}$,
Where $u(t_0) = u'(t_0) = 0$ and $u_0(t) = 0$. Therefore, the equation(3.12) is Hyers-Ulam stable.

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