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# On Application of Fourier Series

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**Abstract**

The Fourier series, the founding principle behind the field of Fourier analysis, is an infinite expansion of a function in terms of sines and cosines. In physics and engineering, expanding functions in terms of sines and cosines is useful because it allows one to more easily manipulate functions that are, for example, discontinuous or simply difficult to represent analytically. In particular, the fields of electronics, quantum mechanics, and electrodynamics all make heavy use of the Fourier series. In this paper we use the concept of Fourier series to solve the non linear Partial Differential Equation.

**Keywords**: Fourier series, Non linear partial differential equation

## 1. INTRODUCTION

The solutions of nonlinear partial differential equations play an important role in the study of many physical phenomena. With the help of solutions, when they exist, the mechanism of complicated physical phenomena and dynamical processes modelled by these nonlinear partial differential equations can be better understood. They can also help to analyze the stability of these solutions and to check numerical analysis for these nonlinear partial differential equations. Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for solutions of nonlinear partial differential equations [7]. Solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, a variety of powerful methods, such as, tanh-sech method [9], extended tanh method [2], hyperbolic function method [13], Jacobi elliptic function expansion method [6], F-expansion method [14], and the First Integral method [3]. The sine-cosine method [9] has been used to solve different types of nonlinear systems of Partial Differential Equations. In this paper, we applied the Fourier series method to solve the nonlinear partial differential equations.

In 1811, Joseph Fourier in his article "Théorie analitique de la chaleur", given the key idea to farm a series with the basic solutions. The orthonormality is the key concept of the Fourier analysis. The general representation of the Fourier series with coefficients $a_0$, $a_n$ and $b_n$ is given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \quad \text{.............................. (1)}$$

The Fourier series are used in the study of periodical movements, acoustics, electrodynamics, optics, thermodynamics and especially in physical spectroscopy as well as in fingerprints recognition and many other technical domains.

The Fourier coefficients are obtained in the following way:

i. by integration of the previous relation between [α, α+2π]

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(t)dt$$

ii. by multiplication of (1) with cosnt and integration between [α, α+2π]

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(t)\cos ntdt$$

iii. by multiplication of (1) with sinnt and integration between [α, α+2π]

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(t)\sin ntdt$$

Any function f(t)can be developed as a Fourier Series

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$, where $a_0$, $a_n$ and $b_n$ are constants, provided :

(i) f(t) is periodic, single-valued and finite.

(ii) f(t) has a finite number of discontinuities in any one period.

(iii)f(t) has at the most a finite number of maxima and minima.

## 2. FOURIER SERIES SOLUTION OF WAVE EQUATION

In this section we represent one of the methods of solving Partial differential equation by the use of Fourier series. In this method consider the homogeneous heat equation defined on a rod of length $2\ell$ with periodic boundary conditions. In mathematical term we must find a solution u=u(x, t) to the problem

$$\begin{aligned} &u_t - ku_{xx} = 0, && -\ell < x < \ell,\ 0 < t < \infty \\ &u(x,0) = f(x), && -\ell \le x \le \ell, \\ &u(-\ell, t) = u(\ell, t), && 0 \le t < \infty \\ &u_x(-\ell, t) = u_x(\ell, t), && 0 \le t < \infty \end{aligned} \qquad \text{...................... (2)}$$

where k > 0 is a constant. The common wisdom is that this mathematical equations model the heat flow u(x, t) is the temperature in a ring 2l, where the initial (t = 0) distribution of temperature in the ring given by the function f. A point in the ring is represented by a point in the interval $[-\ell, \ell]$ where the end points $x=\ell$ and $x=-\ell$ represents same point in the ring. For this reason the mathematical representation of the problem includes the equations $u(-\ell,t) = u(\ell,t)$ and $u_x(-\ell,t) = u_x(\ell,t)$. To obtained a good solution of this problem, it is better if we assume that f is continuous, $f' \in E$, and f satisfies $f(-\ell) = f(\ell)$ and $f'(-\ell) = f(\ell)$. The idea behind this method is first to find all non identical zero solution of the form u(x, t) =X(x)T(t) to the homogenous system

$$\begin{aligned} &u_t - ku_{xx} = 0, && -\ell < x < \ell,\ 0 < t < \infty \\ &u(-\ell,t) = u(\ell,t), && 0 \le t < \infty \\ &u_x(-\ell,t) = u_x(\ell,t), && 0 \le t < \infty \end{aligned} \qquad \text{.......................... (3)}$$

Taking into consideration the system and the fact that u(x,t ) =X(x)T(t). Then

$$u_t(x, t) = X(x)T'(t), \qquad u_{xx}(x, t) = X''(x)T(t)$$

substituting these forms in the equation we obtain

$$X(x)T'(t) - k X''(x)T(t) = 0$$

and thus

$$X(x)T'(t) = k X''(x)T(t)$$

Dividing both side of the equation by kX(x)T(t). We obtain

$$T'(t)/kT(t) = X''(x)/X(x)$$

The expression on the left – hand side is a function of t alone, while the expression on the right-hand side is a function of x. We already know that x and t are independent upon each other , the equation that is given above can hold only if and only if both sided of it is equal to some unknown constant $-\lambda$ for all value of x and t. Thus we may write

$$T'(t)/kT(t) = X''(x)/X(x) = -\lambda$$

Clearly we obtain one pair of differential equations with unknown constant $\lambda$

$$X''(x) + \lambda X(x) = 0$$

$$T'(t) + k\lambda T(t) = 0$$

From those two boundary conditions we derive two conditions. From the boundary condition $u(-\ell,t) = u(\ell,t)$ it follows that for all $t > 0$

$$X(-\ell)T(t) = X(\ell)T(t)$$

There exist two possibilities. Either $T(t)=0$ for all $t \geq 0$ , $X(-\ell)=X(\ell)$.After all, the first possibility leads us to the trivial solution for which we are not interested. So we look to the second condition $X(-\ell)=X(\ell)$. Similarly we obtain the second condition $X'(-\ell)=X'(\ell)$. When we are looking for non trivial solutions of (3) of the form $u(x,t) = X(x)T(t)$ to the equations for X:

$$X''(x)+ \lambda X(x) = 0, \qquad 0 < x < \ell,$$

$$X(-\ell) = X(\ell)$$

$$X'(-\ell) = X'(\ell)$$

$$\text{.............................} \quad (4)$$

We can easily check that values of $\lambda$ for which equation (4) has non trivial solutions are exactly

$$\lambda_n = n^2\pi^2/\ell^2 \qquad n = 0,1,2,...............$$

For $\lambda_0 = 0$ the equation is $X''(x) = 0$ and general solution is

$$X(x) = c_1x+c_2$$

From the condition $X(-\ell) = X(\ell)$we obtain $c_1= 0$, while the condition $X'(-\ell) = X'(\ell)$ is always satisfied. This being so, in this case, the constant function $X(x) = C$ are solutions of (4). For $\lambda_n= n^2\pi^2/\ell^2$, $n \geq \ell$, the equation is

$$X''(x) + (n^2\pi^2/\ell^2)X)(x) = 0$$

General solution has the form of

$$X(x) = c_1\sin(n\pi/\ell)x + c_2\cos(n\pi/\ell)x.$$

Finally, we have two non - trivial linearly independent solution for all $n \in N$ and $\lambda_n= n^2\pi^2/\ell^2$

$$X_n(x) = \cos(n\pi/\ell)x, \qquad X_n^* = \sin(n\pi/\ell)x$$

Every other solution is a linear combination of these two solutions. The values $\lambda_n$ are called the eigenvalues of the problem, and the solution of $X_n$ and $X_n^*$ are called the eigenfunctions associated with the eigen value $\lambda_n$. We also recall that among the eigenvalues we also have $\lambda_0 = 0$, with associated eigenfunction

$$X_0(x) = \ell$$

Now we consider the second equation $T'(t)+k\lambda T(t)=0$. We restrict to our self to $\lambda = \lambda_n = n^2\pi^2/\ell^2$ , n=0,1,2,3................................. for each n there exists non- trivial solution

$$T_n(t) = e^{-k\lambda nt}$$

Every other solution is a constant multiple therefore, so, finally we can summarize, for each $n \in N$ we have pair of non-trivial solution of the form

$$u_n(x,t)= X_n(x)T_n(t) = e^{-k\lambda nt} \cos (n\pi\ell/)x$$

$$u_n^*(x,t)= X_n^*(x)T_n(t) = e^{-k\lambda nt} \sin (n\pi/\ell)x$$

For n = 0 we have the solution

$$u_0(x,t) = X_0(x)T_0(t) = \ell$$

Since the system (4) is homogenous every "infinite linear combination" of the solution is again a solution. So, we have in a sense, an infinity of solution of the general form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-k\lambda_n t}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$$

We must consider the non homogeneous intial condition $u(x,0) = f(x)$, $-\ell \leq x \leq \ell$.

This condition should determine the two sequence of coefficient $\{a_0, a_1, a_2, a_3.........\}$ , $\{b_1, b_2, b_3...........\}$ and

$$f(x) = u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-k\lambda_n t}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$$

.......................................... (5)

We call it as a Fourier series of f on interval $[-\ell\ \ \ell]$. Where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \qquad \text{n=0, 1, 2, 3............}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \qquad \text{n=1, 2, 3..................}$$

## 3. FOURIER SERIES SOLUTION OF NON LINEAR PARTIAL DIFFERENTIAL EQUATION:

We shall now consider the application of the Fourier series method to the non-linear partial differential equation of the form,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=f(x,t) \qquad \text{.................................................. (6)}$$

With the properties:

1. f(x,t) can be represented in the form of

$$f(x,t)=\frac{1}{2}p_0(t)+\sum_{r=1}^{N}[p_r(t)\cos(\frac{2\pi rx}{l})+q_r(t)\sin(\frac{2\pi rx}{l})]$$

.............................................. (7)

2. At t=0 , u(x, t) is representable as

$$u(x,0)=\frac{k_0}{2}+\sum_{r=1}^{m}\ [k_r\cos\frac{2r\pi x}{l}+m_r\sin\frac{2r\pi x}{l}]$$

........................................................ (8)

Where $k_r$ and $m_r$ are constants.

3. U(x,t) is periodic , of periodic $\ell$ .

We assume a solution to (6) of the form

$$u(x,t)=\frac{f_0(t)}{2}+\sum_{r=1}^{\infty}\ [f_r(t)\cos(\frac{2n\pi x}{l})+g_r(t)\sin(\frac{2n\pi x}{l})]$$

...................................................... (9)

This will satisfy the initial condition (8) provided that

$f_r(0)$= kr, r=0,1,2,3........,M & $f_r(0)$=0, r>m

$g_r(0)$=kr, r=0,1,2,3........,M & $g_r(0)$=0, r>m

...................................................... (10)

By differentiation of (9) with respect to t and x, we get,

$$\frac{\partial u}{\partial t}=\frac{1}{2}f_0^{'}(t)+\sum_{r=1}^{\infty}[f_r^{'}(t)\cos(\frac{2\pi rx}{l})+g_r^{'}(t)\sin(\frac{2\pi rx}{l})]$$

$$\frac{\partial x}{\partial x}=\sum_{r=1}^{\infty}[-(\frac{2\pi r}{l})f_r(t)\sin(\frac{2\pi rx}{l})+(\frac{2\pi r}{l})g_r(t)\cos(\frac{2\pi rx}{l})]$$

Let the product of $u(u_x)$ be written in the form

$$u(u_x) = \frac{1}{2}a_0 + \sum_{r=1}^{\infty}[a_r \cos(\frac{2\pi rx}{l}) + b_r \sin(\frac{2\pi rx}{l})]$$

Where the $a_r$ and $b_r$ are related to $f_r$ and $g_r$ in some form. We can obtain this relation by symbolic computation technique, for this firstly we have to define the following operators:

$$c(r) = \cos(\frac{2\pi rx}{l})$$
$$s(r) = \sin(\frac{2\pi rx}{l})$$
$$f(r) = f_r(t)$$
$$g(r) = g_r(t)$$
$$z(r) = \frac{2\pi r}{l}$$

Then u and $u_x$ can be written as

$$u = \frac{1}{2}f_0 + \sum_{r=1}^{\infty}[f(r)c(r) + g(r)c(r)]$$

$$u_x = \sum_{r=1}^{\infty}[-z(r)f(r)s(r) + z(r)g(r)c(r)]$$

By multiplication of u and $u_x$

$$u(u_x) = \left\lfloor \frac{1}{2}f_0 + \sum_{r=1}^{\infty}[f(r)c(r) + g(r)s(r)] \right\rfloor \left\lfloor \sum_{r=1}^{\infty}[-z(r)f(r)s(r) + z(r)g(r)c(r)] \right\rfloor$$

............................................................ (11)

By symbolic computation technique

$$u(u_x) = \frac{1}{2}a_0 + \sum_{r=1}^{\infty}[a_r \cos(\frac{2\pi rx}{l}) + b_r \sin(\frac{2\pi rx}{l})]$$

By substituting the value of $u_t$ and $u(u_x)$ in (6) we have,

$$\frac{1}{2}f_0^{'}(t) + \sum_{r=1}^{\infty}[f_r^{'}(t)\cos(\frac{2\pi rx}{l}) + g_r^{'}(t)\sin(\frac{2\pi rx}{l})] +$$
$$\frac{1}{2}a_0(t) + \sum_{r=1}^{\infty}[a_r(t)\cos(\frac{2\pi rx}{l}) + b_r(t)\sin(\frac{2\pi rx}{l})]$$
$$= \frac{1}{2}p_0(t) + \sum_{r=1}^{N}[p_r(t)\cos(\frac{2\pi rx}{l}) + q_r(t)\sin(\frac{2\pi rx}{l})]$$

..................................................... (12)

Equating coefficient we have

$$\frac{1}{2}f_0^{'}(t)+\frac{1}{2}a_0(t)=\frac{1}{2}p_0(t)$$
$$f_r^{'}(t)+a_r(t)=p_r(t)$$
$$g_r^{'}(t)+b_r(t)=q_r(t)$$
$$r=1,2,3,......\gamma=\max(M,N)$$

........................................................ (13)

The above set of Ordinary Differential Equation's is not sufficient to represent the Partial Differential Equation. This is because the $a_r$ and br are also function of $f_i$ and $g_i$ with i < r. But for r > 2M, the quantities $a_r$ and $b_r$ must contain $f_i$ and $g_i$ with i > M, the following two additional equations will be sufficient to constitute the system of ordinary differential equations to represent (6). These are,

$$f_r^{'}(t)+a_r(t)=0$$
$$g_r^{'}(t)+b_r(t)=0$$
$$r=\gamma+1,\gamma+2,\gamma+3,.................$$

.............................................................. (14)

For r>2M the differential equations are of the form,

$$\frac{df_r}{dt}=0, f_r(0)=0$$
$$\frac{dg_r}{dt}=0, g_r(0)=0\text{ , r>2M}$$

Whose solutions are $f_r(t)\equiv 0, g_r(t)\equiv 0.$

Therefore, the original partial differential equation is transformed into a finite of ordinary differential equations (13) and (14) with the initial conditions (10).

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