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Solving Bessel and Legender equation Using Mellin transform

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Abstract

In this paper the author emphasizes the relation between Mellin and right sided Laplace transform with a view to utilize as a basic tool to solve Bessel and Legender equation.

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1. Introduction

As we know from Mellin transform,

$$M[f(x); s] = F(s) = \int_0^\infty f(x)x^{s-1}dx$$

As per [1] we have

$$L[g(t); s] = \int_{-\infty}^{\infty} g(t)e^{-st}dt = -\int_{\infty}^{0} f(x)x^{s-1}dx$$
$$= M[f(x); s]$$

So the Mellin transform of $f(x)$ is the two-sided Laplace transform of $g(t)$ where $t = -logx$ and it converges absolutely.

Keeping in view of this the author desires to have a relation between Mellin transform and right sided laplace transform as follows. Now

$$L^+[g(t); s] = \int_0^\infty e^{-st}g(t)dt$$

Using $x = e^{-t}$ and $t = -\ln x$, we have $t = 0 \Rightarrow \ln x = 0 = \ln 1 = 2n\pi i = \infty$, no finite value by extending for complex plane.

Again for $t = \infty$, $x = 0$, So

$$L^{+}[g(t); s] = -\int_{\infty}^{0} x^{s} g(\ln x)\frac{1}{x}dx = \int_{0}^{\infty} f(x)x^{s-1}dx = M[f(x); s] = F(s) \quad (1)$$

2. Some important properties of Mellin Transform

Scaling Property:

$$M\{f(at); s\} = \int_{0}^{\infty} f(at)t^{s-1}dt = a^{-s}\int_{0}^{\infty} f(x)x^{s-1}dx = a^{-s}F(s)$$

Multiplication by t^a:

$$M\{t^{a} f(t); s\} = \int_{0}^{\infty} f(t)t^{(s+a)-1}dt = F(s+a)$$

Derivative:

$$M\{\frac{d^k}{ds^k} f(t); s\} = (-1)^k (s-k)_k F(s-k)$$

where

$$(s-k)_k \equiv (s-k)(s-k+1) - - - - - - -(s-1) = \frac{(s-1)!}{(s-k-1)!} = \frac{\Gamma(s)}{\Gamma(s-k)}$$

Derivative Multiplied by Independent Variable:

$$M\{t^k \frac{d^k}{ds^k} f(t); s\} = (-1)^k (s)_k F(s) = (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)} F(s),$$

$$(s)_k = (s)(s+1) - - - - - -(s+k-1)$$

So

$$M(t^2 \frac{d^2 f(t)}{dt^2} + t\frac{df(t)}{dt}; s) = s^2 F(s)$$

3. Main Work

Using (1) and the properties of Mellin Transform, we solve the Bessel equation and Legender equation as well.

Solving Bessel Equation: Solve $t^2 y'' + ty' + (t^2 - \gamma^2)y = 0$

Taking Mellin Transform both the sides, we have

$$M(t^2 y'') + M(ty') + M(t^2 - \gamma^2)y = 0$$

$$\Rightarrow (s+1)sF(s) - sF(s) + F(s+2) - \gamma^2 F(s) = 0$$

Hence

$$s^2 F(s) - \gamma^2 F(s) = -F(s+2)$$
$$\Rightarrow F(s) = \frac{-F(s+2)}{s^2 - \gamma^2}$$

So

$$\frac{F(s)}{F(s+2)} = \frac{-1}{s^2 - \gamma^2}$$

Taking inverse Laplace transform both sides. Hence

$$\begin{aligned} f(t) &= -(L^{+})^{-1}\frac{F(s+2)}{(s)^2 - \gamma^2} \\ &= -(e^{-2t} f(t)) * \left(\frac{1}{\gamma}\sin h\gamma t\right) \\ &= -\frac{1}{\gamma}\int_0^t e^{-2u} f(u) \sin h\gamma (t-u) du \\ &= -\frac{1}{\gamma}\Big[\int_0^t e^{(-2-\gamma)u} e^{t\gamma} f(u) du \\ &\quad - \int_0^t e^{(-2+\gamma)u} e^{-t\gamma} f(u) du\Big] \\ &= -\frac{1}{2\gamma}\Big[-H(t)e^{t\gamma} + k(t)e^{-t\gamma} + A\Big], \end{aligned}$$

Where $A = H(0) - K(0)$.

This may be the solution of Bessel equation with a particular value of $H(t)$, and $K(t)$.

Here the author desires to find the particular solution instead of general solution, satisfying the oscillatory property as well.

Solving Legender Equation:

Solve $(1-t^2)y'' - 2ty' + n(n+1)y = 0$.

Taking transform both sides we have,

$$M((1-t^2)y'') - M(2ty') + M(n(n+1)y) = 0$$

$$M(y'') - M(t^2 y'') - 2M(ty') + n(n+1)M(y) = 0$$

So,

$$(-1)^2(s-2)(s-1)F(s-2) - (s+1)sF(s) - 2sF(s) + n(n+1)F(s) = 0$$

$$F(s) = \frac{-(s^2 - 3s + 2)F(s-2)}{n(n+1) - s^2 - 3s}$$

Hence,

$$\frac{F(s-2)}{F(s)} = \frac{s^2-3s+2+6s-2-n(n+1)}{s^2-3s+2} = 1 + \frac{6s-2-n(n+1)}{(s-1)(s-2)}$$

Now,

$$\frac{6s-2-n(n+1)}{(s-1)(s-2)} = \frac{n^2+n-4}{s-1} + \frac{10-n^2-n}{s-2}$$

So,

$$\frac{F(s-2)}{F(s)} = 1 + \frac{n^2+n-4}{s-1} + \frac{10-n^2-n}{s-2}.$$

Using Taylor's expansion we have,

$$\frac{F(s)-2F'(s)+------}{F(s)} = 1 + \frac{n^2+n-4}{s-1} + \frac{10-n^2-n}{s-2} \tag{2}$$

In particular for $n = 1$,

$$-2\frac{F'(s)}{F(s)} = \frac{-2}{s-1} + \frac{8}{s-2},$$

by neglecting the higher degree terms.

Integrating,

$$\ln F(s) = \ln(s-1) - 4\ln(s-2)$$

So,

$$\begin{aligned} F(s) &= \frac{(s-1)}{s-2)^4} \\ F(s) &= \frac{s-2+1}{(s-2)^4} = \frac{1}{(s-2)^3} + \frac{1}{(s-2)^4} \end{aligned}$$

Taking the inverse Laplace transform both sides,

$$\begin{aligned} f(t) &= (L^+)^{-1}\Big[\frac{1}{(s-2)^3} + \frac{1}{(s-2)^4}\Big] \\ &= e^{2t}\frac{t^2}{2} + e^{2t}\frac{t^3}{6} = (1+2t+2t^2+----)\left(\frac{t^2}{2}+\frac{t^3}{6}\right) \\ &= \frac{t^2}{2} + \frac{7t^3}{6} + ------ \\ &= A + B\left(\frac{1}{2}\right)(3t^2-1) + C\frac{1}{2}(5t^3-3t) + ------- \end{aligned}$$

So,

$$f(t) = Ap_1(t) + Bp_2(t) + Cp_3(t) + -------,$$

for any finite value of A,B,C.

Similarly for $n = 0$, we have the equation (2) becomes

$$\frac{F(s) - 2F'(s) + ------ -}{F(s)} = \frac{-4}{s-1} + \frac{10}{s-2}$$

So,

$$\begin{aligned}
\ln F(s) &= 2\ln(s-1) - 5\ln(s-2) \\
F(s) &= \frac{(s-1)^2}{(s-2)^5} = \frac{1}{(s-2)^3} + \frac{1}{(s-2)^5} + \frac{2}{(s-2)^3} \\
&= e^{2t}\left(\frac{t^2}{2} + \frac{t^4}{24} + \frac{t^3}{3}\right) = ((1 + 2t + 2t^2 + ----)\left(\frac{t^2}{2} + \frac{t^4}{24} + \frac{t^3}{3}\right)
\end{aligned}$$

Hence,

$$f(t) = \frac{t^2}{2} + \frac{t^4}{24} + \frac{4t^3}{3} + ------ -$$

So in particular we may claim this as solution of Legender equation.

The author desires to have particular solution for different values of n.

4. Conclusion

Here the author exhibits a relation between Mellin and Right sided Laplace transform which we can use to solve a differential equation with variable and constant coefficients.Like Bessel's and legender's equation we may apply the process to solve partial differential equation with the relevant initial and boundary conditions.

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