Common Fixed Point Theorem using Weakly Compatible mapping along with (CLRg) Property

Madhu Shrivastava*, Dr. K.Qureshi** and Dr. A.D.Singh**

*TIT Group of Institutions, Bhopal  
**Ret. Additional Director, Bhopal  
**Govt. M.V.M. College, Bhopal

Abstract

We prove some common fixed point theorem for a pair of weakly compatible mapping in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using \( \varphi \)-mapping.

1. INTRODUCTION

The notion of fuzzy set was introduced by Zadeh [18] in 1965. It was developed extensively by many researchers, which also include interesting applications of this theory in different fields. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modelling theory, engineering science, image processing, control theory and communication. In 1975 Kramosil and Michalek [16] introduced the concept of fuzzy metric space. Further George and Veeramani [2] modified the concept of fuzzy metric space. Gregori et al. [32] showed several interesting examples of fuzzy matrices and have also utilized such fuzzy metrics to colour image processing. Subsequently several authors as Mishra et al. [30], Butnariu [12], Badshah and Joshi [31], Singh and Jain [4] gives very important results.

In 2002 Amri and Moutawakil [20] defined E.A. Property for self mapping. In 2011 Sintunavarat and Kumam [38] introduced the “Common limiting range property” which relaxes the conditions of closeness of underlying subspace. Recently Imdad et al. [21] extended the notion of common limit range property of two pairs of self mapping which relaxes the requirement of closeness of subspaces.
In this paper we prove some common fixed point theorem for a pair of weakly compatible mapping in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using \( \phi \)-mapping.

2. PRELIMINARIES

**Definition 2.1** (Schweizer and Sklar [6]). A continuous t-norm is a binary operation \( * \) on \([0, 1] \) satisfying the following conditions:

(i) \( * \) is commutative and associative;

(ii) \( a * 1 = a \) for all \( a \in [0, 1] \);

(iii) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) \( (a, b, c, d \in [0, 1]) \);

(iv) the mapping \( * : [0, 1] \times [0, 1] \to [0, 1] \) is continuous.

**Definition 2.2** (Kramosil and Michalek[16]). A fuzzy metric space is a triple \( (X, M, *) \) where \( X \) is a nonempty set, \( * \) is a continuous t-norm and \( M \) is a fuzzy set on \( X^2 : [0, \infty) \) such that the following axioms hold:

(KM - 1) \( M(x, y, 0) = 0 \) for all \( x, y \in X \);

(KM - 2) \( M(x, y, t) = 1 \) for all \( x, y \in X \) where \( t > 0 \iff x = y \);

(KM - 3) \( M(x, y, t) = M(y, x, t) \) for all \( x, y \in X \);

(KM - 4) \( M(x, y, \cdot) : [0, \infty) \to [0, 1] \) is left continuous for all \( x, y \in X \);

(KM - 5) \( M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \) for all \( x, y, z \in X \) and for all \( s, t > 0 \).

We will refer to these spaces as KM-fuzzy metric spaces.

**Lemma 2.3** (Grabiec [19]). For every \( x, y \in X \), the mapping \( M(x, y, \cdot) \) is non decreasing on \([0, \infty]\).

George and Veeramani [1,2] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek.

**Definition 2.4** (George and Veeramani [1,2]). A fuzzy metric space is a triple \( (X, M, *) \) where \( X \) is a nonempty set, \( * \) is a continuous t-norm and \( M \) is a fuzzy set on \( X^2 : [0, \infty) \) and the following conditions are satisfied for all \( x, y \in X \) and \( t, s > 0 \):

(GV1) \( M(x, y, t) > 0 \);

(GV2) \( M(x, y, t) = 1 \iff x = y \);
Common Fixed Point Theorem using Weakly Compatible mapping along with (CLRg)

(GV3) \( M(x, y, t) = M(y, x, t) \);

(GV4) \( M(x, y, t): (0, \infty) \to [0, 1] \) is continuous;

(GV5) \( M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s) \)

From (GV-1) and (GV-2), it follows that if \( x \neq y \), then \( 0 < M(x, y, t) < 1 \) for all \( t > 0 \).

In what follows, fuzzy metric spaces in the sense of George and Veeramani will be called GV-fuzzy metric spaces.

From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. Several authors have contributed to the development of this theory.

Example 2.5 Let \((X, d)\) be a metric space, \(a \ast b = T_M(a, b)\) and, for all \(x, y \in X\) and \(t > 0\),

\[
M(x, y, t) = \frac{t}{t + d(x, y)}
\]

Then \((X, M, \ast)\) is a GV-fuzzy metric space, called standard fuzzy metric space induced by \((X, d)\).

Definition 2.6 Let \((X, M, \ast)\) be a (KM- or GV-) fuzzy metric space. A sequence \(\{x_n\}\) in \(X\) is said to be convergent to \(x \in X\) if

\[
\lim_{n \to \infty} M(x_n, x, t) = 1, \text{ for all } t > 0
\]

Definition 2.7 Let \((X, M, \ast)\) be a (KM- or GV-) fuzzy metric space. A sequence \(\{x_n\}\) in \(X\) is said to be G-Cauchy sequence if

\[
\lim_{n \to \infty} M(x_n, x_{n + m}, t) = 1, \text{ for all } t > 0, \quad m \in \mathbb{N}
\]

Definition 2.8 A fuzzy metric space \((X, M, \ast)\) is called G-complete if every G-Cauchy sequence converges to a point in \(X\).

Lemma 2.9 (Schweizer and Sklar [6]). If \((X, M, \ast)\) is a KM-fuzzy metric space and \(\{x_n\}, \{y_n\}\) are sequences in \(X\) such that

\[
\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y
\]

Then \(\lim_{n \to \infty} M(x_n, y_n, t) = M(x, y, t)\) for every continuity point \(t\) of \(M(x, y, \cdot)\).
Definition 2.10 (Jungck and Rhoades[13]). Let \( X \) be a nonempty set. Two mappings \( f, g : X \to X \) are said to be weakly compatible if \( fgx = gfx \) for all \( x \) which \( fx = gx \).

In 1995, Subrahmanyam [24] gave a generalization of Jungck’s [14] common fixed point theorem for commuting mappings in the setting of fuzzy metric spaces. Even if in the recent literature weaker conditions of commutativity, as weakly commuting mappings, compatible mappings, \( R \)-weakly commuting mappings, weakly compatible mappings and several authors have been utilizing, the existence of a common fixed point requires some conditions on continuity of the maps, \( G \)-completeness of the space, or containment of ranges.

The concept of E.A. property in metric spaces has been recently introduced by Aamri and El Moutawakil [20].

Definition 2.11 (Aamri and El Moutawakil [20]). Let \( f \) and \( T \) be self-mapping of a metric space \((X,d)\). We say that \( f \) and \( T \) satisfy E.A. property if there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t, \quad \text{for some } t \in X
\]

The concept of E.A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range.

Recently, Mihet [11] proved two common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using E.A. property.

Definition 2.12 Let \( \Phi \) be class of all mappings \( \varphi : [0,1] \to [0,1] \) satisfying the following properties:

\( (\varphi 1) \) \( \varphi \) is continuous and non-decreasing on \([0,1]\);

\( (\varphi 2) \) \( \varphi(x) > x \) for all \( x \in (0,1) \).

Definition 2.13 Suppose that \((X,d)\) is a metric space and \( f,g : X \to X \). Two mappings \( f \) and \( g \) are said to satisfy the common limit in the range of \( g \) property if

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = gx, \quad \text{for some } x \in X.
\]
3. MAIN RESULTS

Theorem 3.1 Let \((X, M, \ast)\) be a KM-fuzzy metric space satisfying the following property:

\[
M(fx, fy, t) \geq \varphi \min\left\{ \frac{M(gx, gy, t) + M(fy, gy, t)}{1 + M(gx, gy, t) + M(fx, gx, t)}, \frac{M(fy, gy, t) + M(fx, gx, t)}{1 + M(fy, gy, t) + M(fx, gx, t)}, \frac{M(fx, gx, t) + M(gx, gy, t)}{1 + M(fx, gx, t) + M(gx, gy, t)}, \frac{M(gx, gy, t) + M(fx, gx, t)}{1 + M(gx, gy, t) + M(fx, gx, t)} \right\}
\]

(1)

for all \(x, y \in X, x \neq y, \exists t > 0: 0 < M(x, y, t) < 1\)

and let \(f\) and \(g\) weakly compatible self mapping of \(X\). If \(f\) and \(g\) satisfy the CLRg property, then \(f\) and \(g\) have a unique common fixed point.

Proof – Since \(f\) and \(g\) satisfy the CLRg property, there exist a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = gx
\]

for some \(x \in X\). Let \(t\) be a continuity point of \((X, M, \ast)\). Then from eq. (1)

\[
M(fx_n, fx, t) \geq \varphi \min\left\{ \frac{M(gx_n, gx, t) + M(fx, gx, t)}{1 + M(fx, gx, t) + M(gx_n, gx, t)}, \frac{M(fx, gx, t) + M(gx_n, gx, t)}{1 + M(fx, gx, t) + M(gx_n, gx, t)} \right\}
\]

(2)

for all \(n \in \mathbb{N}\).

By making \(n \to \infty\), we have

\[
M(gx, fx, t) \geq \varphi \min\left\{ \frac{M(gx, gx, t) + M(fx, gx, t)}{1 + M(fx, gx, t) + M(gx, gx, t)}, \frac{M(fx, gx, t) + M(gx, gx, t)}{1 + M(fx, gx, t) + M(gx, gx, t)} \right\}
\]

\[
= \varphi \min\{1, 1, 1, 1, M(fx, gx, t)\} = \varphi M(fx, gx, t)
\]

(3)

for every \(t > 0\). We claim that \(fx = gx\). If not, then

\[
\exists t_0 > 0: 0 < M(gx, fx, t_0) < 1
\]

(4)

It follows from the condition of (\(\varphi 2\)) that \(\varphi M(gx, fx, t_0) > M(gx, fx, t_0)\),
which is a contradiction. Therefore \( g_x = f_x \).

Next, we let \( z = f_x = g_x \). Since \( f \) and \( g \) are weakly compatible mapping \( f g_x = g f_x \)

Which implies that

\[
fz = fgx = gfx = gz
\]

We claim that \( fz = z \). Assume not, then by (2), its implies that \( 0 < M(fz, z, t_1) < 1 \) for some \( t_1 > 0 \).

By condition \((\varphi 2)\), we have \( \varphi(M(fz, z, t_1)) > M(fz, z, t_1) \). Using condition (1)

Again, we get

\[
M(fz, z, t) = M(fz, fz, t)
\]

\[
\geq \varphi \left\{ \min \left\{ \frac{M(gz, gx, t) + M(fx, gx, t)}{1 + M(fz, gz, t)}, \frac{M(fz, gx, t) + M(fz, gz, t)}{M(gz, gx, t) + M(fz, gz, t)} \right\} \right\}
\]

\[
= \varphi \left\{ \min \left\{ \frac{M(fz, z, t) + M(z, z, t)}{1 + M(fz, z, t)}, \frac{M(fz, z, t) + M(z, z, t)}{M(fz, z, t) + M(z, z, t)} \right\} \right\}
\]

\[
= \varphi \left\{ \min \left\{ 1, M(fz, z, t), M(fz, z, t), 1 \right\} \right\} = \varphi(M(fz, z, t))
\]

for all \( t > 0 \) which is a contradiction. Hence \( fz = z \), that is \( z = fz = gz \).

Therefore \( z \) is a common fixed point of \( f \) and \( g \).

For the uniqueness of a common fixed point, we suppose that \( w \) is another common fixed

point in which \( w \neq z \). It follows from condition (2) that there exist \( t_2 > 0 \) such that

\( 0 < M(w, z, t_2) < 1 \) since \( M(w, z, t_2) \in (0,1) \),

we have

\[
\varphi(M(w, z, t_2)) > M(w, z, t_2)
\]

By virtue of \((\varphi 2)\). From (2), we have

\[
M(z, w, t) = M(fz, fw, t)
\]

\[
\geq \varphi \left\{ \min \left\{ \frac{M(gz, gw, t) + M(fw, gw, t)}{1 + M(fw, gz, t)}, \frac{M(fw, gz, t) + M(fz, gz, t)}{M(gz, gw, t) + M(fw, gw, t)} \right\} \right\}
\]

By virtue of \((\varphi 2)\). From (2), we have
\textbf{Common Fixed Point Theorem using Weakly Compatible mapping along with (CLRg)}

\begin{equation}
\varphi\{\min\{\frac{M(z, w, t) + M(w, w, t)}{1 + M(w, z, t)}, M(w, z, t) + M(z, z, t), M(z, w, t) + M(w, w, t)'}\}
\end{equation}

\begin{equation}
= \varphi\{\min\{1, 1, M(z, w, t), M(z, w, t), 1\}\} = \varphi(M(z, w, t))
\end{equation}

for all \(t > 0\), which is a contradiction.

Therefore, it must be the case that \(w = z\).

which implies that \(f\) and \(g\) have a unique a common fixed point.

This completes the proof.

\textbf{Theorem 3.2} Let \((X, M, \ast)\) be a KM-fuzzy metric space satisfying the following property:

\begin{equation}
M(fx, fy, t) \geq \varphi\{\min\{a. M(gx, gy, t) + b. M(fx, gx, t), c. M(fy, gx, t) + d. M(gx, gy, t),
\ \ \ \ \ \ \ \ \ \ \ e. M(gx, gy, t) + f. M(fx, gy, t),
\ \ \ \ \ \ \ \ \ \ \ e. M(fx, gx, t) + f\}
\end{equation}

\begin{equation}
M(gx, gx, t), M(gy, gy, t)\}\}
\end{equation}

for all \(x, y \in X, x \neq y, \exists \ t > 0: 0 < M(x, y, t) < 1\)

and let \(f\) and \(g\) weakly compatible self mapping of \(X\). If \(f\) and \(g\) satisfy the CLRg property,

then \(f\) and \(g\) have a unique common fixed point.

\textbf{Proof} – Since \(f\) and \(g\) satisfy the CLRg property, there exist a sequence \(\{x_n\}\) in \(X\) such that

\[\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = gx\]

for some \(x \in X\). Let \(t\) be a continuity point of \((X, M, \ast)\).

Then,

\[M(fx_n, fx, t) \geq \varphi\{\min\{a. M(gx_n, gx, t) + b. M(fx_n, gx_n, t), c. M(fx, gx_n, t) + d. M(gx_n, gx, t),
\ \ \ \ \ \ \ \ \ \ \ e. M(gx_n, gx, t) + f. M(fx_n, gx, t),
\ \ \ \ \ \ \ \ \ \ \ e. M(fx_n, gx_n, t) + f\}
\end{equation}

\[M(gx_n, gx_n, t), M(fx, gx, t)\}\} \]
For all \( n \in \mathbb{N} \). By making \( n \to \infty \), we have
\[
= \varphi \{ \min \{ \begin{array}{c}
a. M(gx, gx, t) + b. M(gx, gx, t) \\
a. M(gx, gx, t) + b \\
e. M(gx, gx, t) + f. M(gx, gx, t) \\
e. M(gx, gx, t) + f \\
\end{array} \} \},
\]
\[
= \varphi \{ \min \{ 1, 1, 1, M(fx, gx, t) \} \} = \varphi \{ M(fx, gx, t) \}
\] (9)

for every \( t > 0 \). We claim that \( fx = gx \). If not, then
\[
\exists t_0 > 0: 0 < M(gx, fx, t_0) < 1
\]
It follows from the condition of (\( \varphi 2 \)) that \( \varphi(M(gx, fx, t_0)) > M(gx, fx, t_0) \),
which is a contradiction. Therefore \( gx = fx \).

Next, we let \( z = fx = gx \). Since \( f \) and \( g \) are weakly compatible mapping \( fgx = gfx \)
Which implies that
\[
fx = fgx = gfx = gz
\]
We claim that \( fz = z \). Assume not, then by (8),
its implies that \( 0 < M(fz, z, t_1) < 1 \) for some \( t_1 > 0 \).
By condition \( (\varphi 2) \), we have \( \varphi(M(fz, z, t_1)) > M(fz, z, t_1) \).
using condition (7). Again, we get
\[
M(fz, z, t) = M(fz, fz, t)
\]
\[
\geq \varphi \{ \min \{ \begin{array}{c}
a. M(gz, gx, t) + b. M(fz, gz, t) \\
a. M(fz, gx, t) + b \\
e. M(gz, gx, t) + f. M(fz, gx, t) \\
e. M(fz, gx, t) + f \\
\end{array} \} \},
\]
\[
= \varphi \{ \min \{ \begin{array}{c}
a. M(fz, fz, t) + b. M(fz, fz, t) \\
a. M(fz, fz, t) + b \\
e. M(fz, fz, t) + f. M(fz, fz, t) \\
e. M(fz, fz, t) + f \end{array} \} \},
\]
\[
= \varphi \{ \min \{ 1, M(fz, fz, t), M(fz, fz, t), M(fz, fz, t), 1 \} \}
\]
\[
= \varphi \{ M(fx, gx, t) \}
\] (10)
for all \( t > 0 \) which is a contradiction.
Hence \( fz = z\), that is \( z = fz = gz\). Therefore \( z\) is a common fixed point of \( f\) and \( g\).

For the uniqueness of a common fixed point, we suppose that \( w\) is another common fixed point in which \( w \neq z\). It follows from condition (8) that there exist \( t_2 > 0 \) such that 
\[
0 < M(w, z, t_2) < 1.
\]

Since \( M(w, z, t_2) \in (0,1)\), we have \( \varphi(M(w, z, t_2)) > (M(w, z, t_2)) \). By virtue of \( (\varphi 2)\) from (7), we have 
\[
M(z, w, t) = M(fz, fw, t)
\]
\[
\geq \varphi\left\{ \frac{\text{a. } M(gz, gw, t) + \text{b. } M(fz, gz, t)}{a. M(fz, gw, t) + b} + \frac{\text{c. } M(fw, gz, t) + \text{d. } M(gz, gw, t)}{c. M(fw, gw, t) + d}, \frac{\text{e. } M(gz, gw, t) + f. M(fz, gw, t)}{e. M(fz, gz, t) + f} + \text{M(gz, gw, t), M(fw, gw, t)}} \right\}
\]
\[
= \varphi\left\{ \frac{\text{a. } M(z, w, t) + \text{b. } M(z, z, t)}{a. M(z, w, t) + b} + \frac{\text{c. } M(w, z, t) + \text{d. } M(z, w, t)}{c. M(w, w, t) + d}, \frac{\text{e. } M(z, w, t) + f. M(z, w, t)}{e. M(z, z, t) + f} + \text{M(z, w, t), M(w, w, t)}} \right\}
\]
\[
= \varphi\left\{ \text{min}\{1, M(w, z, t), M(w, z, t), M(z, w, t), 1\} \right\} = \varphi\{M(z, w, t)\} \quad (11)
\]

For all \( t > 0\), which is a contradiction.

Therefore, it must be the case that \( w = z \).

which implies that \( f\) and \( g\) have a unique a common fixed point.

This completes the proof.

**Theorem 3.3** Let \((X, M \ast)\) be a GV-fuzzy metric space satisfying the following property:

\[
M(fx, fy, t) \geq \varphi\left\{ \frac{M(gx, gy, t) + M(fy, gy, t)}{1 + M(fy, gx, t)} + \frac{M(fy, gx, t) + M(fx, gx, t)}{1 + M(fx, gy, t)} \right\}
\]

\[
M(fx, gy, t) + M(fy, gx, t)\]

\[
M(fx, gx, t) + M(fy, gy, t), M(gx, gy, t), M(fy, gy, t)) \right\}, \frac{M(fx, gy, t) + M(fy, gx, t)}{M(fx, gx, t) + M(fy, gy, t)}, M(gx, gy, t), M(fy, gy, t)) \right\} \right\} \quad (12)
\]
for all \(x, y \in X\), \(\exists t > 0\) and let \(f\) and \(g\) weakly compatible self mapping of \(X\).

If \(f\) and \(g\) satisfy the CLRg property, then \(f\) and \(g\) have a unique common fixed point.

**Proof** – Since \(f\) and \(g\) satisfy the CLRg property, there exist a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = g x
\]

(13)

For some \(x \in X\). Let \(t\) be a continuity point of \((X, M, *)\). Then

\[
M(f x_n, f x, t) \geq \varphi \{ \min \left\{ \frac{M(g x_n, g x, t) + M(f x, g x, t)}{1 + M(f x, g x, t)}, \frac{M(f x_n, g x_n, t) + M(f x_n, g x_n, t)}{1 + M(f x_n, g x_n, t)}, \frac{M(f x_n, g x, t) + M(f x, g x, t)}{1 + M(f x, g x, t)} \right\} \}
\]

For all \(n \in \mathbb{N}\). By making \(n \to \infty\), we have

\[
M(g x, f x, t) \geq \varphi \{ \min \left\{ \frac{M(g x, f x, t) + M(f x, g x, t)}{1 + M(f x, g x, t)}, \frac{M(g x, g x, t) + M(f x, f x, t)}{1 + M(f x, f x, t)}, \frac{M(g x, g x, t) + M(f x, g x, t)}{1 + M(f x, g x, t)} \right\} \}
\]

(14)

\[
M(g x, f x, t) \geq \varphi \{ \min \{1, 1, 1, M(f x, g x, t)\} \} = \varphi \{M(f x, g x, t)\}
\]

for every \(t > 0\). We claim that \(fx = gx\).

If not, then from (GV-1) and (GV-2)

\[
0 < M(g x, f x, t) < 1, \text{ for all } t > 0
\]

(15)

It follows from the condition of (\(\varphi 2\)) that \(\varphi (M(g x, f x, t)) > M(g x, f x, t)\), which is a contradiction. Therefore \(gx = fx\).

Next, we let \(z = fx = gx\). since \(f\) and \(g\) are weakly compatible mapping \(fgx = gfx\)

Which implies that

\[
fz = fgx = gfx = gz
\]

We claim that \(fz = z\). Assume not, then by (GV-1) and (GV-2), its implies that \(0 < M(fz, z, t) < 1\) for all \(t > 0\). By condition \(\varphi 2\), we have \(\varphi (M(fz, z, t)) > M(fz, z, t)\).
using condition (12), again, we get

\[
\frac{M(fz, z, t) = M(fz, fx, t)}{\geq \phi \{ \min \{ \frac{M(gz, gx, t) + M(fz, gx, t)}{1 + M(fz, gz, t)} \cdot M(fz, gz, t) + M(fz, gz, t) \}, \frac{M(fz, gz, t) + M(fz, gz, t)}{M(fz, gz, t) + M(fz, gx, t)} \}}
\]

\[
= \phi \{ \min \{ \frac{M(fz, z, t) + M(z, z, t)}{1 + M(z, fz, t)} \cdot M(z, fz, t) + M(z, z, t) \}, \frac{M(fz, z, t) + M(z, fz, t)}{M(fz, fz, t) + M(z, z, t)} \} \}
\]

\[
= \phi \{ \min \{1, 1, M(fz, z, t), M(fz, z, t), 1 \} \} = \phi (M(fz, z, t))
\]

(16)

For all \( t > 0 \) which is a contradiction. Hence \( fz = z \), that is \( z = fz = gz \).

Therefore \( z \) is a common fixed point of \( f \) and \( g \).

For the uniqueness of a common fixed point, we suppose that \( w \) is another common fixed point of \( f \) and \( g \) in which \( w \neq z \). It follows from condition (15) that there exist \( t > 0 \) such that

\( 0 < M(w, z, t) < 1 \). since \( M(w, z, t) \in (0, 1) \), we have \( \phi (M(w, z, t)) > M(w, z, t) \).

By virtue of (\( \phi 2 \)),

\[
M(z, w, t) = M(fz, fw, t)
\]

\[
\geq \phi \{ \min \{ \frac{M(gz, gw, t) + M(fw, gw, t)}{1 + M(fw, gz, t)} \cdot M(fw, gz, t) + M(fw, gz, t) \}, \frac{M(fz, gw, t) + M(fw, gz, t)}{M(fz, gz, t) + M(fw, gw, t)} \} \}
\]

\[
= \phi \{ \min \{ \frac{M(z, w, t) + M(w, w, t)}{1 + M(w, z, t)} \cdot M(w, z, t) + M(w, w, t) \}, \frac{M(z, w, t) + M(w, w, t)}{M(z, z, t) + M(w, w, t)} \} \}
\]

\[
= \phi \{ \min \{1, 1, M(z, w, t), M(z, w, t), 1 \} \} = \phi (M(z, w, t))
\]

(17)
for all \( t > 0 \), which is a contradiction.

Therefore, it must be the case that \( w = z \).

which implies that \( f \) and \( g \) have a unique a common fixed point.

This completes the proof.

**Theorem 3.4** Let \((X, M\ast)\) be a GV-fuzzy metric space satisfying the following property :

\[
M(\text{fx}, \text{fy}, t) \geq \varphi\{\min\{\begin{array}{ll}
\text{(a)} & M(\text{gx}, \text{gy}, t) + b. M(\text{fx}, \text{gx}, t), \\
\text{(b)} & M(\text{fy}, \text{gx}, t) + d. M(\text{gx}, \text{gy}, t) \\
\text{(c)} & M(\text{fx}, \text{gy}, t) + d
\end{array}\}, \begin{array}{ll}
\text{(d)} & M(\text{gx}, \text{gy}, t) + f. M(\text{fx}, \text{gy}, t) \\
\text{(e)} & M(\text{fx}, \text{gx}, t) + f
\end{array}\}, \begin{array}{ll}
\text{M}(\text{gx}, \text{gy}, t), M(\text{fx}, \text{gy}, t)\} \}
\]

(18)

for all \( x, y \in X, \exists t > 0 \) and let \( f \) and \( g \) weakly compatible self mapping of \( X \).

If \( f \) and \( g \) satisfy the CLRg property, then \( f \) and \( g \) have a unique common fixed point.

**Proof** – Since \( f \) and \( g \) satisfy the CLRg property, there exist a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} \text{fx}_n = \lim_{n \to \infty} \text{gx}_n = \text{gx}
\]

(19)

for some \( x \in X \). Let \( t \) be a continuity point of \((X, M\ast)\). Then

\[
M(\text{fx}_n, \text{fx}, t) \geq \varphi\{\min\{\begin{array}{ll}
\text{(a)} & M(\text{gx}_n, \text{gx}, t) + b. M(\text{fx}_n, \text{gx}_n, t), \\
\text{(b)} & M(\text{fx}, \text{gx}_n, t) + d. M(\text{gx}_n, \text{gx}, t) \\
\text{(c)} & M(\text{fx}, \text{gx}, t) + d
\end{array}\}, \begin{array}{ll}
\text{(d)} & M(\text{gx}_n, \text{gx}, t) + f. M(\text{fx}_n, \text{gx}, t) \\
\text{(e)} & M(\text{fx}, \text{gx}_n, t) + f
\end{array}\}, \begin{array}{ll}
\text{M}(\text{gx}_n, \text{gx}, t), M(\text{fx}, \text{gx}, t)\} \}
\]

for all \( n \in \mathbb{N} \). By making \( n \to \infty \), we have

\[
= \varphi\{\min\{\begin{array}{ll}
\text{(a)} & M(\text{gx}, \text{gx}, t) + b. M(\text{gx}, \text{gx}, t), \\
\text{(b)} & M(\text{fx}, \text{gx}, t) + d. M(\text{gx}, \text{gx}, t) \\
\text{(c)} & M(\text{fx}, \text{gx}, t) + d
\end{array}\}, \begin{array}{ll}
\text{(d)} & M(\text{gx}, \text{gx}, t) + f. M(\text{gx}, \text{gx}, t) \\
\text{(e)} & M(\text{fx}, \text{gx}, t) + f
\end{array}\}, \begin{array}{ll}
\text{M}(\text{gx}, \text{gx}, t), M(\text{fx}, \text{gx}, t)\} \}
\]

(20)
For every \( t > 0 \).

We claim that \( f(x) = g(x) \). If not, than from (GV-1) and (GV-2)

\[
0 < M(g(x), f(x), t) < 1, \text{ for all } t > 0
\]

(21)

It follows from the condition of \((\varphi 2)\) that \( \varphi(M(g(x), f(x), t)) > M(g(x), f(x), t) \), which is a contradiction. Therefore \( g(x) = f(x) \).

Next, we let \( z = f(x) = g(x) \). Since \( f \) and \( g \) are weakly compatible mapping \( f(g(x)) = g(f(x)) \).

Which implies that

\[
f(z) = f(g(x)) = g(f(x)) = g(z)
\]

We claim that \( f(z) = z \). Assume not, then by (GV-1) and (GV-2), it implies that

\[
0 < M(f(z), z, t) < 1, \text{ for some } t > 0.
\]

By condition \((\varphi 2)\), we have

\[
\varphi(M(f(z), z, t)) > M(f(z), z, t).
\]

Again, we get

\[
\varphi\{\min\{1, M(f(x), f(z), t), M(f(x), f(z), t), M(f(x), f(z), t), 1]\} = \varphi\{M(f(x), g(x))\}
\]

(22)

for all \( t > 0 \) which is a contradiction.

Hence \( f(z) = z \), that is \( z = f(z) = g(z) \). Therefore \( z \) is a common fixed point of \( f \) and \( g \).

For the uniqueness of a common fixed point, we suppose that \( w \) is another common fixed point in which \( w \neq z \).
It follows from condition (21) that there exist $t > 0$ such that

$0 < M(w, z, t) < 1$. Since $M(w, z, t) \in (0, 1)$, we have $\varphi(M(w, z, t)) > M(w, z, t)$

By virtue of $(\varphi 2)$,

$$M(z, w, t) = M(fz, fw, t)$$

$$\geq \varphi \{ \min \left\{ \frac{a. M(gz, gw, t) + b. M(fz, gz, t)}{a. M(fz, gw, t) + b}, \frac{c. M(fw, gz, t) + d. M(gz, gw, t)}{c. M(fw, gw, t) + d}, \frac{e. M(gz, gw, t) + f. M(fz, gw, t)}{e. M(fz, gz, t) + f} \right\}, \{g, M(gz, wt), M(fw, gw, t)\}\}$$

$$= \varphi \{ \min \left\{ \frac{a. M(z, w, t) + b. M(z, z, t)}{a. M(z, w, t) + b}, \frac{c. M(w, z, t) + d. M(z, w, t)}{c. M(w, w, t) + d}, \frac{e. M(z, w, t) + f. M(z, z, t)}{e. M(z, z, t) + f} \right\}, \{M(z, w, t), M(w, w, t)\}\}$$

$$= \varphi \{ \min \{1, M(w, z, t), M(w, z, t), M(z, w, t), 1\}\} = \varphi \{M(z, w, t)\} \quad (23)$$

for all $t > 0$, which is a contradiction. Therefore, it must be the case that $w = z$

which implies that $f$ and $g$ have a unique common fixed point.

This completes the proof.

**Corollary 3.5** Let $(X, M, *)$ be a KM-fuzzy metric space satisfying the following property:

For all $x, y \in X, x \neq y$, $\exists t > 0 : 0 < M(x, y, t) < 1$,

and let $f, g$ be weakly compatible self mapping of $X$, such that for some $\varphi \in \emptyset$,

$$M(fx, fy, t) \geq \varphi \{ \min \left\{ \frac{a. M(gx, gy, t) + b. M(fx, gx, t)}{a. M(fx, gy, t) + b}, \frac{c. M(fy, gx, t) + d. M(gx, gy, t)}{c. M(fy, gy, t) + d}, \frac{e. M(gx, gy, t) + f. M(fx, gy, t)}{e. M(fx, gx, t) + f} \right\}, \{gx, gy, t\}, M(fy, gy, t)\}\}$$

If $f$ and $g$ satisfy E.A property and the range of $g$ is a closed subspace of $X$ then $f$ and $g$

have a unique common fixed point.
Proof- Since f and g satisfy E.A. property there exist a sequence \( \{x_n\} \) in X such that

\[
\lim_{n \to \infty} f_{x_n} = \lim_{n \to \infty} g_{x_n} = u, \quad \text{for some } u \in X.
\]

It follows from \( gX \) being closed subspace of X that \( u = gx \) for some \( x \in X \) and then f and g satisfy (CLRg) property, and then we easily have a common fixed point.

**Corollary 3.6** Let \((X, M, \ast)\) be a GV-fuzzy metric space and let \( f, g \) be weakly compatible self mapping of \( X \), such that for some \( \varphi \in \emptyset \),

\[
M(fx, fy, t) \geq \varphi \left\{ \min \left\{ \begin{array}{l}
a.M(gx, gy, t) + b.M(fx, gx, t) + c.M(fy, gx, t) + d.M(gx, gy, t) \\
e.M(fx, gx, t) + f.M(fx, gy, t)
\end{array} \right\} \right.
\]

If for all \( x, y \in X, t > 0, f \) and \( g \) satisfy E.A property and the range of \( g \) is a closed subspace of \( X \), then \( f \) and \( g \) have a unique common fixed point.

Proof- Since f and g satisfy E.A. property there exist a sequence \( \{x_n\} \) in X such that

\[
\lim_{n \to \infty} f_{x_n} = \lim_{n \to \infty} g_{x_n} = u, \quad \text{for some } u \in X.
\]

It follows from \( g(X) \) being closed subspace of X that there exist for some \( x \in X \) then \( u = gx \) and then f and g satisfy (CLRg) property, and then we easily have a common fixed point.

**REFERENCES**


