Mathematical modeling of blood flow in an inclined tapered artery under MHD effect through porous medium

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Abstract

The purpose of this work is to study the effect of blood flow in an inclined tapered artery under MHD effect through porous medium. Blood is considered as an electrically conducting Newtonian fluid. In this model, the analytical expressions for the volumetric flow rate, pressure gradient, wall shearing stress, and velocity profile have been derived. The problem is described by the usual MHD equations along with suitable boundary conditions. There is also a noticeable effect of permeability on the volumetric flow rate. Some of the found results show that the flow patterns in non-tapered region (ξ = 0), converging region (ξ < 0), and diverging region (ξ > 0), are effectively influenced by the presence of magnetic field and change in leaning of artery.

Keywords: Inclined tapered artery, MHD equations, Stenotic region.

AMS Subject Classification (1991): 760z05, 92c35

1. Introduction:
Now-a-days many people are suffering from cardiovascular disease such as
Atherosclerosis (medically called stenosis) which is causes of death of people. A blockage by atherosclerosis, which is progressive vascular disease that causes collection of fatty substances, cholesterol, fibrin, calcium, and cellular waste. It is also known as plaque inside the wall of arteries. It leads to the narrowing of the internal space of the arteries and effects as carotid artery (major blood vessels in the neck that supply blood to the brain). When the plaque hardens and narrows down the arteries completely, then the blood and oxygen supply to the brain are reduced. This is one of the major contributing factors to strokes. This leads to block of brain function and the death of people.

This is considerable evidence that vascular fluid dynamics plays an important role in the development and progression of arterial stenosis. The theory presented so far is enough for uniform visco-elastic tubes but, is inadequate for real arteries because they are not uniform. They suffer both continuous variation in cross sectional area and dispensability from repeated branching. If we take tapered tube the normal blood flow is disturbed. The main disadvantage is using a tapered geometry however, is the much greater energy losses which may leads to diminished blood flow through the tapered grafts. It is important therefore these losses are quantified and taken account in the design of tapered grafts.


If we apply a magnetic field to a moving electrically conducting liquid, it induces electric and magnetic fields. The interplay of these fields produces a body force known as Lorentz force which has a tendency to oppose the movement of the liquid. For the blood flow in arteries with arterial diseases like arteriosclerosis, influence of magnetic field may be utilized as a blood pump in carrying out cardiac operations, and in addition to this, the effect of vessels tapering together with the shape of stenosis on the flow character seem to be equally important and deserve special attention.


In this study, we are analyzing the characteristics of the blood flow through an inclined tapered porous artery with mild stenosis under the influence of an inclined magnetic field. This study can play a big role in the conclusion of axial velocity, shear stress and fluid acceleration in porous medium. This study is also useful for evaluating the role of porosity. The study is carried out by employing approximate analytical method.

2. Mathematical formulation:
We consider one-dimensional study, laminar and fully developed flow of blood through inclined tapered artery, with the presence of mild stenosis. It is assume that the formation of stenosis which is symmetrical about the axis, but non symmetrical with respect to radial co-ordinates, and it depends upon the height and location of the constriction, formed at the innermost wall and the axial wall. It is assumed that the wall of the tapered tube is rigid. There is no loss of generally in considering a rigid artery as due to the formation of a stenosis, the elasticity of the arterial wall gets reduced. Further, the artery length is assumed to be large enough as compared to its radius, so that the entrance and special wall effects can be neglected.

The geometry of an arterial non-symmetrical stenosis in a tapering wall can be expressed (Mekheiner and Kothari, 2008) as:

\[
R(z) = \begin{cases} 
    h(z)[1 - \eta(b^{n-1}(z-a) - (z-a)^2)], & a \leq z \leq a + b \\
    h(z), & \text{otherwise}
\end{cases}
\]  

with \( h(z) = h_0 + \xi z \);  

\[ (1) \]

where, \( R(z) \) is the radius of the stenosed portion of arterial segment and \( h(z) \) is the radius of tapered arterial segment in the stenotic region, \( h_0 \) is the radius of the non tapered artery in the non stenotic region, \( \xi \) is the tapering parameter, \( b \) is taking the length of the stenosis, and \( n \geq 2 \) is being a parameter determining the shape constriction and referred to as a shape parameter.
Here we are using the parameter $\eta$ which is given by:

$$\eta = \frac{\delta}{h_{0}b^{n}} \left( \frac{n^{-1}}{n - 1} \right)$$  \hspace{1cm} (3)

where, $\delta$ denotes the maximum height of the stenosis to be found at:

$$z = a + \frac{b}{n^{1/(n-1)}}$$  \hspace{1cm} (4)

**Figure (1):** Geometry an inclined stenosed artery with axially non-symmetrical stenosis
Here the body fluid is assumed to behave as a Newtonian fluid (Schlitchting and Gerstein 2004). The equation (as obtained from Navier-Stokes equation of motion for various fluids) describing the steady flow of Newtonian fluid is given by (Schlitchting and Gerstein 2004):
\[
\frac{\partial p}{\partial r} = 0 \quad (5)
\]
\[
\frac{\partial p}{\partial \theta} = 0 \quad (6)
\]

3. Analytical Solution of the problem:
As per the published literature and available physiological data, blood flow in the neighborhood of the vessel wall can be considered as Newtonian, if the shear rate of blood is high enough. However, the shear rate is very small towards the center of the artery (circular tube), the non-Newtonian behavior of blood is more evident (Mishra et. al 2007).

The steady flow of blood through the cylindrical artery inclined at an angle \( \alpha \) can be written as follows:
\[
\nabla \cdot \mathbf{V} = 0 \quad \text{(7)}
\]
\[
\rho \, \mathbf{F} = -\nabla p + \mu \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right\} - \frac{\mu V}{\kappa} + J \times \mathbf{B} \quad \text{(8)}
\]

Boundary conditions for the problem stated above may be listed as:
\[
u = 0 \quad \text{at} \quad r = R(z) \quad \text{(artery wall)}
\]
\[
\frac{\partial u}{\partial r} = 0, \quad \text{at} \quad r = 0 \quad \text{(9)}
\]
\[
u = \text{finite} \quad \text{at} \quad r = 0
\]

The above equation (8) transformed in the term as:
\[ \rho \sin \alpha g = -\frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] - \frac{\mu u}{\kappa} - \sigma B_o^2 u \]  

(10)

where, \( g \) is the acceleration due to gravity, \( \mu \) is the viscosity of blood, \( \kappa \) is the permeability of porous medium, \( \rho \) is the fluid density, \( \alpha \) is the inclination of an artery, \( B_o \) is an applied magnetic field with an inclination \( \theta \).

The non-dimensional variables are:

\[ r = \frac{\bar{r}}{R_o}, \quad z = \frac{\bar{z}}{R}, \quad R = \frac{\bar{R}}{R_o}, \quad v = \frac{b \bar{v}}{\delta u_o}, \quad u = \frac{\bar{u}}{u_o}, \]

\[ p = \frac{\bar{p} R_o^2}{b \mu u_o}, \quad R_e = \frac{\bar{\mu} u_o R_o}{\mu}, \quad \kappa = \frac{\kappa}{R_o^2}, \quad F_r = \frac{u_o^2}{g R_o}, \quad M = \frac{\sigma R_o^2 B_o^2}{\mu} \]  

(11)

Substituting (11) in (10), we can get a dimensionless form for (8) as follows:

\[ \left( \frac{R_o}{F_r} \right) \sin \alpha = -\frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] - \left( \frac{1}{\kappa} + M^2 \cos^2 \theta \right) u \]  

(12)

As the flow is steady and axisymmetric, let the solution for \( u(r,t) \) and \( p \) be set in the forms:

\[ u(r,t) = \bar{u}(r), \quad -\frac{\partial p}{\partial z} = P \]  

(13)

where, \( P \) is a constant. Substitute equation (13) in (12), we can have a second order ordinary differential equation as follows:

\[ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \beta^2 u = \frac{R_e}{F_r} \sin \alpha - P \]  

(14)

where \( \beta^2 = \left( \frac{\kappa}{\kappa + M^2 \cos^2 \theta} \right) \)

The solution of second order differential equation (14) can be written as follows:

\[ u(r) = \beta^2 \left( \frac{R_o}{F_r} \sin \alpha - P \right) \left( \frac{J_0(\beta r)}{J_0(\beta h)} - 1 \right) \]  

(15)

where, \( J_0 \) is the modified Bessel’s function of the zero order

The volumetric flow rate \( Q \) of fluid is the stenotic region is given by:

\[ Q = 2\pi R_o \int_0^h ru(r) dr \]  

(16)

Substitute \( u(r) \) from (15) into (16) an then integrating with respect to \( r \), we obtain:
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\[
Q = \frac{1}{\beta^2} \left( \frac{R_e}{F_r} \sin \alpha - P \right) \left( -\frac{h^2}{2} + \frac{h}{\beta J_1(\beta h)} \right)
\]

(17)

where, \(J_1\) is the modified Bessel’s function of the order one.

The wall shear stress is defined by:

\[
\tau_r = -\mu(r) \frac{du}{dr} \bigg|_{r=h}
\]

(18)

which, on using (15) gives:

\[
\tau_r = \frac{\mu}{\beta} \left( \frac{R_e}{F_r} \sin \alpha - P \right) \left( \frac{J_1(\beta h)}{J_1(\beta h)} \right)
\]

(19)

4. Result and Discussion:

Most of the theoretical result such as permeability, inclination angle of artery (\(\alpha\)), the inclination of magnetic field (\(\theta\)), wall shear stress, shear stress at the stenosis throat, axial velocity, and volumetric flow rate are obtained in the numerical analysis with computational illustrations. Out of these results, only the numerical solution of wall shear stress and relative local pressure gradient are shown graphically for different values of Reynolds number \(R_e\), Froude number \(F_r\) and Hartmann number \(M\) for better understanding of the problem.

All graphs are plotted by using mathematical software MATLAB, for the value \(R_e = 0.1, F_r = 0.1, M = 2, 3\) and \(4, \sigma = \frac{b}{a} = 0, b = 1\), stenosis height \(\delta = 0\) to \(20\) shape parameter \(n = 2, 6,\) and \(11\), \(\xi = \tan \phi = 0.002, 0, -0.002\).

Figure (3) depicts that the variation of wall stress \(\tau\), with shape parameter \(n\) and permeability. It is evident that the wall stress decreases with increases of permeability parameter.

\[
\text{Figure (3): Effect of permeability on the shearing stress for different values of shape parameter (n) and permeability (k).}
\]
The wall shear stress attains maximum value in case of radically symmetric stenosis \((n = 2)\) and starts diminishing as stenosis losing its symmetric i. e. stenosis shape parameter \(n\) becoming larger.

By the inspection of the figure 4, it can be noticed that the converging region \((\xi < 0)\), the stress will be more as compared to the diverging region \((\xi > 0)\) and non tapered region \((\xi = 0)\) that is wall shear stress increases with the increase in tapering angle \((\xi)\).

**Figure (4):** The result of artery inclination \((\alpha)\) on the shearing stress for different values of tapering angle \((\xi)\)

**Figure (5):** Variation of shearing stress at the stenosis throat \((\delta)\) with different inclination \((\alpha)\) for different values of permeability \((\kappa)\).
Figure 6: Variation of axial velocity \( u \) with \( z \) and height of stenosis \( \delta \) for different values of permeability \( \kappa \).

It is quite interesting to observe from the figure (5) that as the variation of shearing stress at the stenosis throat for different values of inclination of artery. It has been noticed that with the increase of inclination \( \alpha \), the shearing stress \( \tau \) increases. It is also observed that the variation of permeability affects the stress inversely. Figure (6) illustrates that variation of axial velocity \( u \) with \( z \) and the height of stenosis \( \delta \) for different value permeability \( \kappa \). It is clear that the axial velocity possesses reverse behavior on either side of the centre line of the artery. It shows that an increase in axial velocity with the increase of permeability \( \kappa \).

Figure (7): Variation of axial velocity \( u \) with magnetic field for different values of tapering angle \( \xi \).
In figure (7), with the increase of magnetic field, the axial velocity shows a reverse behavior. It is observed that with the increase of magnetic field, the curve representing the axial flow velocity do shift towards the origin for a converging region, while they shift away from the origin for a non tapered and diverging tapered artery.

It is quite interesting to observe from the figure (8) that as the argumentation in the axial velocity shows the remarkable changes with the inclination of artery. It has been observed that with the increase of inclination artery, the curve representing the axial flow velocities does shift towards the origin.

Figure (9): Axial velocity with the inclination of magnetic field (θ), for $R_s = 0.1$, $F_s = 0.1$, $\kappa = 2$, $M = 2$, and $\alpha = \pi/3$
Figure (10): Variation of volumetric flow rate with the tapering angle ($\xi$), for different values of ($M$) and for fixed values of $R_e = 0.1$, $F_r = 0.1$, $\kappa = 2$, and $\alpha = \pi / 3$

Figure (9) reveals that the variation of axial velocity with the inclination of magnetic field ($\theta$). It has been observed that the volumetric flow rate will increase and will be more for diverging region as compared to converging region. It is seen that the increase of inclination of magnetic field, the curve does shift towards the origin. Figure (10) shows an increasing behavior of volumetric flow rate from converging to diverging region. The volumetric flow rate for an inclined artery will decrease with the Hartmann number for the fixed values of Reynolds number Reynolds number ($R_e$) = 0.1, Froude number ($F_r$) = 0.1, permeability ($\kappa$) = 2, and artery inclination ($\alpha$) = $\pi / 3$.

The volumetric flow rate for an inclined artery will be greater in diverging region ($\xi > 0$) as compared to converging region ($\xi < 0$).
Figure (11) : Variation of volumetric flow rate with the angle of inclination $\alpha$ of artery for converging region ($\xi < 0$), diverging region ($\xi > 0$) and non-tapered region ($\xi = 0$)

Figure (12) : Variation of volumetric flow rate with the angle of magnetic field $\theta$ for converging region ($\xi < 0$), diverging region ($\xi > 0$) and non-tapered region ($\xi = 0$)

Figure (11) illustrates that the variation of volumetric flow rate with the angle of inclination $\alpha$ of artery for converging region ($\xi < 0$), diverging region ($\xi > 0$) and non-tapered region ($\xi = 0$), It is seen that the increases of inclination $\alpha$, the volumetric flow rate decreases. Figure (12) depicts that the variation of volumetric flow rate with the angle of magnetic field $\theta$ will increases for all converging region ($\xi < 0$), diverging region ($\xi > 0$) and non-tapered region ($\xi = 0$).
Conclusion:
In the present investigation, we have developed a mathematical model of blood flow in an inclined tapered artery, with the presence of mild stenosis under the MHD effect through porous medium. Analytical expressions of flow variables are obtained and variations of shear stresses at stenotic wall resistance to flow are shown graphically. This investigation can play a vital role in the determination of axial velocity, shear stress, and permeability in particular situations. Since, this study has been carried out for a situation when the human body is subjected to an external magnetic field. The study is also useful for evaluating the roll of porosity. It is observed that magnetic field reduces the flow characteristics amazingly. Also the height of stenosis significantly affects the shearing stress, wall shear stress and axial velocity. This investigation may be helpful for the practitioners to treat the hypertension patient through magnetic therapy and to understand the flow of blood under stenotic conditions.

REFERENCES:


