THEORY OF MICROPOLAR THERMOELASTIC MATERIALS WITH VOIDS

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Abstract. A present article is a theoretical study of micropolar thermoelastic materials with voids. The basic equations and constitutive equations are derived from Goodman and Cowin [1] theory. The considered void is distributed partially throughout the medium, thus the voids volume interactions with micro-polarity of the materials are not considered for the derivation of field equation.

Keywords: Micropolar, Voids, Entropy, Stress tensor, Deformation tensor, Heat flux vector

1. Introduction

The mechanical behaviour of material with voids containing microscopic components was not described by classical theory of elasticity. Eringen [2-3] introduced a theory of thermo-microstretch elastic solid and established a uniqueness theorem for mixed initial boundary value problem; he also introduced a linear theory of micropolar elasticity which takes into account the micro-structural motions. Goodman and Cowin [1] introduced the concept of distributed body in the context of granular materials, in which matrix or skeletal is elastic and interstices are voids, in their theory the mass density has a decomposition of matrix materials, and volume fractional field, but the effect of micro-structure is not considered. Nunziato and Cowin [4] proposed a theory to describe the properties of homogeneous elastic materials with voids following some basic ideas on granular materials found by Goodman and Cowin [1]. In Cowin and Nunziato [5], a theory of linear elastic materials with voids was developed, in which a distributed body is written as the product of two field, the material density field and volume fractional field. Passarella [6] investigated some results on micropolar thermoelastic materials; he derived the constitutive equations and field equation for micropolar porous thermoelastic materials. In Ciarletta et al. [7], the plane waves and vibrations in the theory of micropolar thermoelastic materials with voids are discussed. Lianngenga and Lalawmpuia [8] discussed micropolar elastic materials containing voids. Marin [9] discussed some basic theorems in elastostatic of
micropolar materials with voids. Aouadi [10] derived the equations of linear theory of thermoelastic diffusion in porous medium based on the concept of volume fraction. In this paper, the wave phenomenon is investigated which depends not only on the microstructure of materials but also on the existence of voids and thermal in the considered medium. The present work clarifies the difference of voids inclusion from the inclusion of porous in the micropolar medium. The presence of voids is known to affect the estimation of the physical and mechanical properties of the composite. We assume that the voids contain nothing of mechanical or energetic significant, and the distribution of voids is partially to the medium. By employing Helmholtz free energy, the present theory is proposed for the determination of equilibrated stress vector and intrinsic equilibrated body force as well as stress and couple stress tensor. Section 2 is derivation of basic equations. In section 3, constitutive equations are derived from the free energy. Section 4 discusses linear theory and establishes field equation for the considered medium. Section 5 is conclusions.

2. Derivation of Basic Equations

Definition: A material is called porous material if the void volume distribution is completely throughout the medium. And a material is called voids material if the void volume distribution is partially throughout the medium [8].

Let us consider the region $B_0$ and its boundary $\partial B_0$ in three dimensional vector spaces occupied by Micropolar thermoelastic materials with voids in a reference configuration. And let $B$ and $\partial B$ be the region and its boundary in the current configuration. We shall assumed that the voids volume distribution is partially throughout the body $B$. And let $\nu$ denotes the density of the matrix material, and $\psi$ be the volume distribution function, then mass density $\rho$ of the region has taken the form

$$\rho = \rho_0 \nu$$

where, $0 < \psi \leq 1$, $\nu = \nu (X ; t)$, $\psi = \psi (X ; t)$, $\rho = \rho (X ; t)$. At a reference state, the density or the voids distribution can be different, thus the equation (1) may take the form

$$\rho_0 = \psi_0 \nu_0$$

where, $\rho_0$, $\psi_0$ and $\nu_0$ are the equivalent functions to $\rho$, $\psi$ and $\nu$ in the reference configuration, respectively.

Following Aouadi [10], the energy balance of Micropolar thermoelastic materials with voids has the form

$$\int_B \rho \left( \dot{u}_i \ddot{u}_j + j_{ij} \dot{\Phi} \ddot{\Phi} + \chi \dot{\psi} \ddot{\psi} + \dot{E} \right) \, dV = \int_B \rho \left( f_i \dot{u}_i + l_i \dot{\Phi}_i + l \dot{\psi} + q \right) \, dV + \int_{\partial B} \left( T_i \dot{u}_i + M_i \dot{\Phi}_i + h_i \dot{\psi} + q_i \right) \, dS$$

This equation is valid for every time and every part of $B$; where $V$ and $S$ are volume and surface area of the region $B$ respectively; $l$, $q$, $q_i$ are external equilibrated body force, strength of internal heat source and heat-flux vector; $f_i$, $l_i$ are components of
external forces and couple forces per unit mass; \( u_i, \phi_i, \psi \) are component of displacement vector, micro-rotation and volume fraction field; \( T_i, M_i, j_{ij} \) are components of stress vector, couple stress vector and micro-inertia tensor; \( h_i, \chi \) are equilibrated stress, equilibrated inertia; \( \rho, E \) are mass density and internal energy. The superposed dot implies partial derivatives with respect to time variable.

From equation (3) by using invariance under superposed rigid body motion for every time and every part of \( B \), we obtain

\[
\int_B \left( \rho \ddot{u}_i \right) \ dV = \int_B \left( \rho f_i \right) \ dV + \int_{\partial B} T_i \ dS
\]

and

\[
\int_B \left( \rho j_{ij} \ddot{\psi}_i \right) \ dV = \int_B \left( \rho l_i \right) \ dV + \int_{\partial B} M_i \ dS
\]

thus, from equation (4) we obtain (see [10])

\[
t_{i,j} + \rho f_i = \rho \ddot{u}_i
\]

the equation (6) is a linearised form of balance of momentum. Similarly from equation (5) we can obtain (see [9])

\[
m_{i,j} + \epsilon_{jrs} t_{ls} + \rho l_i = \rho j_{ij} \ddot{\psi}_j
\]

where \( t_{ij} \) and \( m_{ij} \) are stress tensor and couple stress tensor; \( \epsilon \) is an alternating symbol and the subscript preceded by coma implies covariant derivative. The equation (7) is also linearised form of balance of angular momentum.

In the view of equation (6) and (7) the equation (3) becomes

\[
\frac{d}{dt} \int_B \left( \rho E + \frac{\rho g \psi^2}{2} \right) \ dV = \int_B \left( t_{ij} \dot{E}_{ji} + m_{ij} \dot{F}_{ij} + \rho l \dot{\psi} + pq + q_{i,i} \right) \ dV + \int_{\partial B} h_i \dot{\psi} \ dS
\]

where \( h_i \) is equilibrated stress associated with the surface in the deformation; \( E_{ij} \) and \( F_{ij} \) are deformation and wryness tensor respectively.

From equation (8), employing divergence theorem we obtain local balance of energy

\[
\rho \dot{E} = t_{ij} \dot{E}_{ji} + m_{ij} \dot{F}_{ij} - g \dot{\psi} + h_i \dot{\psi}_{,i} - q_{i,i} + pq
\]

where \( g \) is introduced from the following balance of equilibrated stress (see Nunziato and Cowin [4])

\[
h_{i,i} + g + \rho l = \rho \chi \dot{\psi}
\]

And entropy inequality

\[
\rho \dot{\eta} \geq \left( \frac{pq}{T} \right) - \left( \frac{q_x}{T} \right)_x
\]

where, \( \eta \) is entropy change per unit mass; \( g, h_i \) are intrinsic equilibrated body force, equilibrated stress vector; \( T \) is absolute temperature. The symbols are described as in
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the referential configuration, although \( \rho \) and \( \phi \) are bulk mass density and change in volume fraction from reference configuration.

3. Constitutive Equations

Let us include \( q_i \) in the sets of independent constitutive variables. The medium is assumed to have same temperature at each and every points, thus the set of constitutive variables will be \( \Sigma = \{ E_{ij}, F_{ij}, \psi, \dot{\psi}_i, T, q_i \} \). Thus, the constitutive equations will take the forms

\[
t_{ij} = t_{ij}(\Sigma); \quad m_{ij} = m_{ij}(\Sigma); \quad g = g(\Sigma); \quad h_i = h_i(\Sigma); \quad \dot{q}_i = \dot{q}_i(\Sigma); \quad \eta = \eta(\Sigma)
\]

(12)

Following Passarella [6]

\[
\frac{\delta \dot{q}_i}{\delta q_j} = 0 \quad \text{for} \ i \neq j
\]

(13)

Introducing Helmholtz free energy function \( \Psi = \Psi(\Sigma) \) defined as

\[
\Psi = E - \eta T
\]

(14)

from equations (9), (11) and (14), we obtain

\[
\rho \dot{\Psi} = t_{ij} \dot{E}_{ij} + m_{ij} \dot{F}_{ij} - g \dot{\psi} + h_i \dot{\psi}_i + q_{i,i} - \rho \dot{\eta} T - \rho \dot{T} \eta + pq
\]

(15)

\[
\rho \dot{\Psi} \leq t_{ij} \dot{E}_{ij} + m_{ij} \dot{F}_{ij} - g \dot{\psi} + h_i \dot{\psi}_i - \rho \dot{T} \eta
\]

(16)

now, differentiating \( \Psi = \Psi(\Sigma) \) we get

\[
\rho \dot{\Psi} = \rho \left( \frac{\delta \psi}{\delta E_{ij}} \right) \dot{E}_{ij} + \rho \left( \frac{\delta \psi}{\delta F_{ij}} \right) \dot{F}_{ij} + \rho \left( \frac{\delta \psi}{\delta \psi_i} \right) \dot{\psi}_i + \rho \left( \frac{\delta \psi}{\delta T} \right) \dot{T} + \rho \left( \frac{\delta \psi}{\delta q_i} \right) \dot{q}_i
\]

(17)

from equations (15-17) we obtain

\[
\left( t_{ij} - \rho \frac{\delta \psi}{\delta E_{ij}} \right) \dot{E}_{ij} + \left( m_{ij} - \rho \frac{\delta \psi}{\delta F_{ij}} \right) \dot{F}_{ij} - \left( g + \rho \frac{\delta \psi}{\delta \psi_i} \right) \dot{\psi} + \left( h_i - \rho \frac{\delta \psi}{\delta \psi_i} \right) \dot{\psi}_i + \left( \rho \frac{\delta \psi}{\delta T} + \rho \eta \right) \dot{T} + q_{i,i} - \rho \frac{\delta \psi}{\delta q_i} \dot{q}_i - \rho \dot{\eta} T + pq = 0
\]

(18)

\[
\left( t_{ij} - \rho \frac{\delta \psi}{\delta E_{ij}} \right) \dot{E}_{ij} + \left( m_{ij} - \rho \frac{\delta \psi}{\delta F_{ij}} \right) \dot{F}_{ij} - \left( g + \rho \frac{\delta \psi}{\delta \psi_i} \right) \dot{\psi} + \left( h_i - \rho \frac{\delta \psi}{\delta \psi_i} \right) \dot{\psi}_i + \left( \rho \frac{\delta \psi}{\delta T} + \rho \eta \right) \dot{T} - \rho \frac{\delta \psi}{\delta q_i} \dot{q}_i - \rho \dot{\eta} T \leq 0
\]

(19)

If we suppose that \( t_{ij} \) is independent of \( \dot{E}_{ij} \), \( m_{ij} \) is independent of \( \dot{F}_{ij} \), \( g \) is independent of \( \dot{\psi} \), \( h_i \) is independent of \( \dot{\psi}_i \) and \( \eta \) is independent of \( \dot{T} \), then the above equations (18-19) will be satisfied for arbitrary \( \dot{E}_{ij}, \dot{M}_{ij}, \dot{\psi}_i, \dot{\psi}_i, \dot{T} \) provided

\[
t_{ij} = \rho \frac{\delta \psi}{\delta E_{ij}}; \quad m_{ij} = \rho \frac{\delta \psi}{\delta F_{ij}}; \quad g = -\rho \frac{\delta \psi}{\delta \psi_i}; \quad h_i = \rho \frac{\delta \psi}{\delta \psi_i}; \quad \rho \eta = -\rho \frac{\delta \psi}{\delta T}
\]

(20)
these equations are the constitutive equations for Micropolar thermoelastic materials with voids. In this equation (20), we deduce that $g$ is given as derivative of Helmholtz free energy, although some of the possibly desirable features of visco-elasticity are lost. It is because the function $g$ and $q_i$ still involve $\dot{\psi}$ and this lead to the behaviour of visco-elasticity (Nunziato and Cowin [4]). For simplicity we shall omit $\dot{\psi}$ from the sets of constitutive variables. Similarly, from equations (18-19) the entropy inequalities and equation of energy balance will respectively become

$$\rho \frac{\delta v}{\delta q_i} \dot{q}_i \leq 0 ;$$

$$\rho \left( \frac{\delta v}{\delta q_i} \dot{q}_i + \eta \dot{T} - q \right) - q_{i,i} = 0$$

4. Linear theory

Now define the deformation tensor and wryness tensor as

$$E_{ij} = u_{ji} - \epsilon_{ijk} \phi_k,$$  \hspace{1cm} (23)

$$F_{ij} = \phi_{ji},$$  \hspace{1cm} (24)

where, $u_i$ and $\phi_i$ are displacement vector and micro-rotation vector respectively, $\epsilon_{ijk}$ is an alternating symbol. Let us recall the linearised forms of local balance equations (6), (7) and (10) for the considered medium as follows

Balance of momentum: $t_{ij,j} + \rho f_i = \rho \ddot{u}_i$ \hspace{1cm} (25)

Balance of angular momentum: $m_{ij,j} + \epsilon_{ijk} t_{jk} + \rho l_i = \rho f_{ij} \ddot{\phi}_i$ \hspace{1cm} (26)

Balance of equilibrated stress: $h_{ij,i} + g + \rho l = \rho \chi \ddot{\psi}_i$ \hspace{1cm} (27)

where, $f_{ij}$ is micro-inertia tensor, $\chi$ and $l$ are equilibrated inertia and external body force, $f_i$ and $l_i$ are body force and body couple, $\rho$ is mass density. In Cowin and Nunziato [5], the inertia coefficient $\chi$ is allowed to depend on coordinate axes and time, but for simplicity we shall follow Goodman and Cowin [1] and assumed it to be constant.

Let $\theta$ be the temperature measured from the absolute temperature $\theta_0 (> 0)$ in a reference state (i.e. $\theta = T - \theta_0$) and take $\Psi(0) = \Psi_0 = 0$. Expanding $\Psi = \Psi(E_{ij}, F_{ij}, \psi, \psi_i, \theta, q_i)$ by quadratic approximation method, we obtain

$$\rho \Psi = A_{ij} E_{ij} + B_{ij} F_{ij} + A \psi + A_i \psi, i + B \theta + B_i q_i + \frac{1}{2} A_{ijkl} E_{ij} E_{kl} + \frac{1}{2} B_{ijkl} F_{ij} F_{kl} + \frac{1}{2} \chi \psi^2 + \frac{1}{2} C_{ijkl} \psi_j \psi_k + \frac{1}{2} D_{ijkl} q_j q_k + G_{ij} E_{ij} F_k + H_{ij} E_{ij} \psi + L_{ijkl} E_{ij} \psi_k + F_{ij} F_{ij} \psi + J_{ijkl} F_{ij} \psi_k + K_{ij} E_{ij} \psi + L_{ijkl} E_{ij} q_k + M_{ij} \psi_i \psi_i + \xi \theta + N_i \theta q_i + O_{ij} E_{ij} + Q_{ijkl} F_{jkl} + P \psi + R_{ijkl} \psi_i + S_i \theta + T_{ijkl} \theta i - q_i$$

(28)

similarly, expanding $\dot{q}_i = \dot{\psi}_i$ by the same method, we obtain

$$\dot{q}_i = \frac{1}{\tau} \left( O_i + P_{ijkl} E_{jkl} + Q_{ijkl} F_{jkl} + P_i \psi + R_{ijkl} \psi_i + S_i \theta + T_{ijkl} \theta i - q_i \right)$$

(29)

where, $\tau$ be the thermal relaxation time for the considered materials.
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At a reference state, the material is free from stress and couple stress with zero entropy, we shall have the following restriction for the above coefficients

\[ A_{ij} = B_{ij} = A = A_{i} = B = B_{i} = C_{i} = I_{ijk} = L_{ijk} = N_{i} = 0 \]  
\[ O_{i} = P_{ijk} = P_{i} = Q_{ijk} = R_{ij} = S_{i} = 0 \]

(30) \hspace{1cm} (31)

Since the medium contain voids (which is partially distributed all over the body). We shall assume that the coupling effect of \( F_{ij} \) and \( \psi \) is no more in the free energy of the medium, thus \( F_{ij} \) is taken zero. And if the material is homogeneous, we shall have

\[ A_{ijkl} = A_{klij}; B_{ijkl} = B_{klij}; C_{ij} = C_{ji}; D_{ij} = D_{ji} \]

(32)

using the above relations the equations (28-29) will take the form

\[ \rho \Psi = \frac{1}{2} A_{ijkl} E_{ij} E_{kl} + \frac{1}{2} B_{ijkl} F_{ij} F_{kl} + \frac{1}{2} \zeta \psi^2 + \frac{1}{2} C_{ijkl} \psi_{,i} \psi_{,j} - \frac{1}{2} D_{ijkl} \psi_{,i} \psi_{,j} + G_{ijkl} E_{ij} F_{kl} + H_{ij} E_{ij} \psi + H_{ijkl} E_{ij} \psi_{,k} + I_{ij} E_{ij} \theta + K_{ij} F_{ij} \theta + L_{ij} \psi \psi_{,i} + \xi \theta + M_{ij} \psi_{,i} \theta + M_{ijkl} \psi_{,i} \psi_{,j} \]

(33)

and

\[ \dot{q}_{i} = \frac{1}{\tau} \left( T_{ij} \theta_{,i} - q_{i} \right) \]

(34)

equation (34) is equation of heat conduction, \( \tau = 0 \) represent Faurier law of heat conduction, and \( T_{ij} \) is thermal conductivity. And at a reference state, \( \theta = \theta_{0} \) and the material has no heat flux rate, thus from equation (22) we can obtain

\[ \rho (\theta_{0} \dot{\theta} - q) - q_{i, i} = 0 \]

(35)

this is a linear form of energy equation. Now, using equation (25-27), the constitutive equation (20) will become

\[ t_{ij} = A_{ijkl} E_{ij} + G_{ijkl} F_{kl} + H_{ij} \psi + H_{ijkl} \psi_{,k} \]

(36)

\[ m_{ij} = B_{ijkl} F_{ij} + G_{ijkl} E_{ij} + K_{ij} \theta \]

(37)

\[ g = -\zeta \psi - H_{ij} E_{ij} - L_{i} \psi_{,i} - \xi \theta \]

(38)

\[ h_{i} = C_{ij} \psi_{,i} + H_{ijk} E_{ij} + L_{i} \psi + M_{i} \theta \]

(39)

\[ \rho \eta = -I_{ij} E_{ij} - K_{ij} F_{ij} - \xi \psi - M_{ij} \psi_{,i} + d \theta \]

(40)

Again, using the operator \( \left( 1 + \tau \frac{\delta}{\delta t} \right) \) to the equation (35) and by equations (23-24), (34) and (40) we get

\[ T_{ij} \theta_{,ij} + \theta_{0} \left( 1 + \tau \frac{\delta}{\delta t} \right) \left[ I_{ij} \left( \ddot{u}_{ij} - \epsilon_{ijk} \dot{\phi}_{,k} \right) + K_{ij} \dot{\phi}_{,ij} + \xi \dot{\psi} + M_{ij} \psi_{,i} - d \dot{\theta} + \frac{\rho q}{\theta_{0}} \right] = 0 \]

(41)

Keeping mind equations (23-24), and putting equations (36-40) in to the equation (25-27), we obtain the coupled system

\[ A_{ijkl} \left( u_{i,j,k} - \epsilon_{kth} \dot{\phi}_{h,i} \right) + G_{ijkl} \dot{\phi}_{t,k} + H_{ij} \psi_{,j} + H_{ijkl} \psi_{,kj} + I_{ij} \theta_{,j} + \rho f_{i} \]

\[ = \rho \ddot{u}_{i} \]

(42)
These equations (41-44) forms a complete set of linear coupled system of field equations in the unknown \( u_i \), \( \phi_i \) and \( \psi_i \). The constitutive coefficients become (see Passarella [6])

\[
H_{ij} = s \delta_{ij}
\]

\[
C_{ij} = a \delta_{ij}
\]

\[
j_{ij} = j \delta_{ij}
\]

\[
l_{ij} = -(3\lambda + 2\mu + \kappa) \epsilon \delta_{ij} = -m \delta_{ij}
\]

\[
T_{ij} = k \delta_{ij}
\]

\[
B_{ijkl} = a \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}
\]

\[
A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + (\mu + \kappa) \delta_{ij} \delta_{kl} + \mu \delta_{ij} \delta_{kl}
\]

\[
G_{ijkl} = H_{ij} = K_{ij} = L_i = M_i = 0
\]

then, the constitutive equation becomes

\[
t_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij} + \kappa(u_{j,i} - \epsilon_{ijk} \phi_k) + s \psi \delta_{ij} + m \theta \delta_{ij}
\]

\[
m_{ij} = \alpha \phi_{k,k} \delta_{ij} + \beta \phi_{j,i} + \gamma \phi_{i,j}
\]

\[
g = -s u_{k,k} - \zeta \psi - \xi \theta
\]

\[
h_i = \alpha \psi_i
\]

\[
\rho \eta = m u_{i,i} - \zeta \psi + d \theta
\]

therefore, the field equations (41-44) will take the form

\[
k \theta_{,kk} + \theta_0 \left(1 + \frac{\tau \delta}{\delta t}\right) \left[-m \dot{u}_{k,k} + \zeta \dot{\psi} + M_i \dot{\psi}_i - d \dot{\theta} + \frac{\rho \eta}{\theta_0}\right] = 0
\]

\[
(\mu + \kappa) u_{i,k,k} + (\lambda + \mu) u_{k,k,i} + \kappa \epsilon_{ijk} \phi_{k,j} + s \psi_{,i} - m \theta_{,i} + \rho f_i = \rho \ddot{u}_i
\]

\[
\gamma \phi_{i,k,k} + (\alpha + \beta) \phi_{k,k,i} + \kappa \epsilon_{ijk} \phi_{k,j} - 2 \kappa \phi_i + \rho l_i = \rho j \ddot{\phi}_i
\]

\[
\alpha \psi_{,kk} - s u_{k,k} - \zeta \psi - \xi \theta + \rho l = \rho \chi \ddot{\psi}
\]

the equations (59-62) are linear isothermal and homogeneous isotropic field equations for micropolar thermoelastic materials with voids (see also Ciarletta et al. [7]). Where \( j, \chi \) are micro-inertia and equilibrated inertia; \( \kappa \) and \( \epsilon \) are thermal conductivity and linear thermal expansion; \( \tau, d, k, \epsilon \) are thermal parameters; \( \lambda, \mu \) are Lame's...
parameters; \( \kappa, \alpha, \beta, \gamma, j \) are micropolar parameters; \( a, \zeta, \xi, s, \chi \) are Void's parameters.

Following Eringen [2], if we consider the non-negative strain energy density function, we shall have the following relations

\[
(3\lambda + 2\mu + \kappa)\zeta > 3s^2; \quad 2\mu + \kappa > 0; \quad 3\alpha + \beta + \gamma > 0; \quad \gamma \pm \beta > 0; \quad \kappa, \alpha, \zeta, k > 0
\]  
(63)

if we assumed that \( d > 0 \) and \( \tau > 0 \), it is clear that the equation (59) will become heat equation of hyperbolic type, propagating with a finite speed of \( (k/\theta_0 \tau d)^{\frac{1}{2}} \).

5. Conclusions

The difference of voids presence from the presence of porous in the micropolar medium has been discussed. In case of voids present in the micropolar thermoelastic medium the interaction of voids and micropolarity are neglected. Thus, the constitutive equations and field equations are derived for the micropolar thermoelastic materials with voids. This interactions may not be negligible in case of porous medium, see Passarella [6].

References