International Journal of Mathematical Education.
ISSN 0973-6948 Volume 8, Number 1 (2018), pp. 1–4

http://www.ripublication.com/

A Deduction of the Sine and Cosine Series using the Laplace Transform

Julio Cesar Mohnsam
COMAT-CINAT
IF Sul, Campus Pelotas-RS, Praça 20 de Setembro 455,
Pelotas-RS, CEP 96.015-360, Brazil.
E-mail: prof.juliomatfis@hotmail.com

Ricardo Capiberibe Nunes
E.E. Amlio de Caravalho Bas
Campo Grande-MS - 79115-020, Brazil.
E-mail: capiberibe@gmail.com

Abstract

In this article, we will present an alternative proof of the Sine and Cosine series using the Laplace Transform. Looking for different tests in Mathematics is very motivating for students in Mathematics because it allows to use other ideas and concepts, and sometimes can bring simpler solutions.

AMS subject classification: 44A10; 41A58.
Keywords: Infinite series; Laplace transform; Sine; Cosine.

1. Introduction

Recently the Binomial Theorem was demonstrated using the Laplace Transform methodology (see [2]). In this work we will prove that the infinite Sine and Cosine series can also be proved using the Laplace transform together with the integration by parts. To arrive at this first we will present the analytical definitions of the Sine and Cosine function.

Proposition 1.1. The Sine function by the Taylor series around $x = 0$:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ; \forall x \in \mathbb{R} \tag{1.1}$$

Proposition 1.2. The Cosine function is defined by the Taylor formula around $x = 0$:

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ; \forall x \in \mathbb{R} \tag{1.2}$$

2. Preliminaries

Now let us recall some important concepts for the deductions of said series.

Proposition 2.1. The Laplace Transformation of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, which is a unilateral transformation defined by

$$\mathcal{L}[f(t)](s) = \int_0^{\infty} f(t) e^{-st} \mathrm{dt} \tag{2.3}$$

where s is a complex number parameter [1].

Example: Laplace Transform of Cosine.

$$\mathcal{L}[\cos(at)](s) = \int_0^{\infty} \cos(at) e^{-st} \mathrm{dt} = \frac{s}{s^2 + a^2}$$

Example: Laplace Transform of Sine.

$$\mathcal{L}[\sin(at)](s) = \int_0^{\infty} \sin(at) e^{-st} \mathrm{dt} = \frac{a}{s^2 + a^2}$$

Proposition 2.2. The inverse Laplace transform is given by the following complex integral,

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds, \tag{2.4}$$

where is a real number so that the contour path of integration is in the region of convergence of $F(s)$ [1].

Lemma 2.3. (Integration by Parts). Let f and g be real functions which are continuous on the closed interval $[a, b]$. Let f and g have primitives F and G respectively on $[a, b]$.

Then:

$$\int_a^b f(t) G(t) \mathrm{dt} = [F(t) G(t)]_a^b - \int_a^b F(t) g(t) \mathrm{dt} \tag{2.5}$$

3. Deduction of series using the Laplace Transform

3.1. Calculation of the Cosine Series

$$\mathcal{L}[\cos t](s) = \int_0^{\infty} \cos t e^{-st} \mathrm{dt}$$

Using integration by parts we have:

$$u = \cos t,\ \mathrm{du} = -\sin t\mathrm{dx};$$

$$\mathrm{dv} = e^{-st}\mathrm{dt},\ v = \frac{-e^{-st}}{s}.$$

$$\mathcal{L}[\cos t](s) = \left[\cos t\left(\frac{-e^{-st}}{s}\right)\right]_0^\infty - \frac{1}{s}\int_0^\infty (\sin t)e^{-st}\mathrm{dt} = \frac{1}{s} - \frac{1}{s}\int_0^\infty (\sin t)e^{-st}\mathrm{dt}$$

$$\mathcal{L}[\cos t](s) = \frac{1}{s} - \frac{1}{s}\left\{\left[\sin t\left(\frac{-e^{-st}}{s}\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty (\cos t)e^{-st}\mathrm{dt}\right\} = \frac{1}{s} - \frac{1}{s^2}\int_0^\infty (\cos t)e^{-st}\mathrm{dt}$$

$$\vdots$$

$$\mathcal{L}[\cos t](s) = \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{1}{s^7} + \cdots = \sum_{k=0}^\infty (-1)^k \frac{1}{s^{2k+1}}$$

Now let's apply the inverse Laplace transform

$$\begin{aligned}\cos t = \mathcal{L}^{-1}\{\mathcal{L}[\cos t]\}(t) &= \mathcal{L}^{-1}\left[\sum_{k=0}^\infty (-1)^k \frac{1}{s^{2k+1}}\right](t) \\ &= \sum_{k=0}^\infty \mathcal{L}^{-1}\left[(-1)^k \frac{1}{s^{2k+1}}\right](t) = \sum_{k=0}^\infty (-1)^k \frac{t^{2k}}{(2k)!}\end{aligned}$$

3.2. Calculation of the Sine Series

$$\mathcal{L}[\sin t](s) = \int_0^\infty \sin t e^{-st}\mathrm{dt}$$

Using integration by parts we have:

$$u = \sin t,\ \mathrm{du} = \cos t\mathrm{dx};\mathrm{dv} = e^{-st}\mathrm{dt},\ v = \frac{-e^{-st}}{s}.$$

$$\mathcal{L}[\sin t](s) = \left[\sin t\left(\frac{-e^{-st}}{s}\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty (\cos t)e^{-st}\mathrm{dt} = 0 + \frac{1}{s}\mathcal{L}[\cos t](s) = \frac{1}{s}\mathcal{L}[\cos t](s)$$

From the previous integration in the cosine part we have

$$\begin{aligned}\mathcal{L}[\sin t](s) = \frac{1}{s}\mathcal{L}[\cos t](s) &= \frac{1}{s}\left(\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{1}{s^7} + \cdots\right) \\ &= \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \cdots = \sum_{k=0}^\infty (-1)^k \frac{1}{s^{2k+2}}\end{aligned}$$

Now let's apply the inverse Laplace transform

$$\begin{aligned}\sin t = \mathcal{L}^{-1}\{\mathcal{L}[\sin t]\}(t) &= \mathcal{L}^{-1}\left[\sum_{k=0}^{\infty}(-1)^k\frac{1}{s^{2k+2}}\right](t)\\ &= \sum_{k=0}^{\infty}(-1)^k\mathcal{L}^{-1}\left[\frac{1}{s^{2k+2}}\right](t) = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}\end{aligned}$$

Acknowledgement

We would like to thank the teachers of the Mathematics Coordination of the IF-SUL Campus Pelotas for their comments.

References

[1] S. Lipschutz, *Mathematical Handbook of Formulas and Tables. Schaum's Outline Series*, McGraw-Hill, 3rd ed, p. 183.

[2] K. Kataria, *An Alternate Proof of the Binomial Theorem*, The American Mathematical Monthly, Vol 123 - Month 11. 2016, p. 940.