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Differential Transform Method for Obtaining Positive Solutions for Two-Point Nonlinear Boundary Value Problems

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Abstract

In this paper, we present a fast and accurate numerical scheme for the solution of a class of two-point nonlinear boundary value problems that have at least one positive solution. The differential transform method is applied to construct the numerical-analytical solution. The new approach provides the solution in the form of a rapidly convergent series with easily computable components and not at grid points. The scheme is tested on three problems. The results demonstrate reliability and efficiency of the algorithm developed.

AMS Subject Classification:
Keywords:

1. Introduction

In the present paper we investigate the following nonlinear boundary value problem:

$$\begin{aligned} &u''(x) + \mu F(x, u(x)) = 0, \quad 0 < x < 1, \\ &u(0) = 0, \quad u(1) = c, \end{aligned} \tag{1.1}$$

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where $\mu \geq 0$, c is a constant, and $F : (0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and is not identically zero on any subset of its domain.

This problem has been considered by Agarwal *et al.* [1–3], Ha and Lee [4], and O'Regan [5,6] using different techniques. The approaches followed in these works were focused on theoretical proofs about the existence of positive solutions of boundary value problems. It was proven that under suitable conditions on $F(x, u(x))$, the two-point boundary value problem (1.1) has at least one positive solution for μ belonging to a compatible interval [3]. Theorems which discuss the conditions for the existence of positive solutions of BVPs are contained in a book by Agarwal [1] and in two other books by O'Regan [5,6]. No numerical methods are contained in these books for solving such problems. However, in literature numerical techniques such as Adomian's decomposition method [7] and Weighted residual method [8] do exist. For more details about the theory of this problem, see [1,3,5,6].

In this paper, we will use a differential transform method to solve (1.1). The differential transform method was first applied in the engineering domain by [9]. In general, the differential transform method is applied to the solution of electric circuit problems. The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method enables us to obtain a series solution by means of an iterative procedure, which is the main advantage of this technique. The method is well addressed in [9,10].

This paper is organized as follows: In Section 2, we describe the differential transform method. In Section 3, the method is implemented to three examples, and finally conclusions are given in Section 4.

2. Differential Transform Method

In this section we shall give the basic theorems of the one-dimensional differential transform method. For more details, see the mentioned references.

The differential transform of the kth derivative of a function $f(x)$ is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2.1}$$

and the inverse differential transform of $F(k)$ is defined as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{2.2}$$

In real applications, the function $f(x)$ is expressed by a finite series and Eq. (2.2) can be written as

$$f(x) = \sum_{k=0}^{n} F(k)(x - x_0)^k. \tag{2.3}$$

The following theorems that can be deduced from Eqs. (2.1) and (2.2) are given below [10]:

Theorem 2.1. If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.

Theorem 2.2. If $f(x) = ag(x)$, then $F(k) = aG(k)$, where a is a constant.

Theorem 2.3. If $f(x) = \dfrac{d^m g(x)}{dx^m}$, then $F(k) = \dfrac{(m+k)!}{k!}G(m+k)$.

Theorem 2.4. If $f(x) = g(x)h(x)$, then $F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.

Theorem 2.5. If $f(x) = x^n$, then $F(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$

Theorem 2.6. If $f(x) = g_1(x)g_2(x)\cdots g_{n-1}(x)g_n(x)$, then

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_2(k_1)G_1(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}).$$

3. Numerical Applications

Example 3.1. We first consider the following boundary value problem [7]

$$u''(x) + 2(u'(x))^2 + 8u(x) = 0, \quad 0 < x < 1, \tag{3.1}$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 0. \tag{3.2}$$

One can see that the differential transform of Eq. (3.1) can be evaluated by using Theorems 2.1, 2.2, 2.3 and 2.4 as follows:

$$U(k+2) = -\frac{1}{(k+1)(k+2)}\left[2\sum_{l=0}^{k}(l+1)(k-l+1)U(l+1)U(k-l+1) + 8U(k)\right] \tag{3.3}$$

and we apply the differential transform at $x_0 = 0$, therefore, the boundary conditions are transformed as follows:

$$U(0) = 0, \quad U(1) = a, \tag{3.4}$$

where according to Eq. (2.1), $a = u'(0)$.

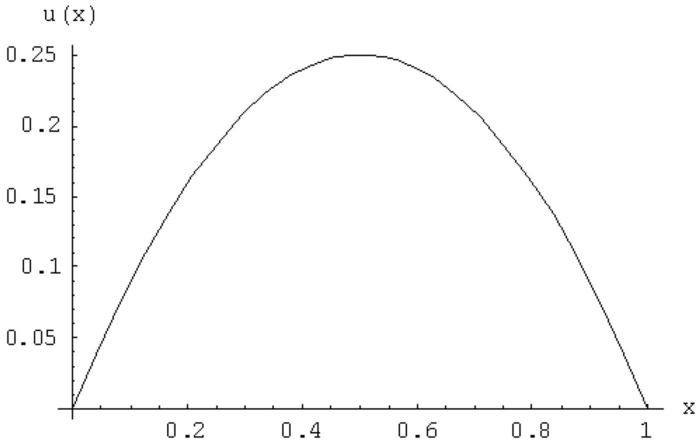

Figure 1: The graph of the exact solution (3.6).

Using Eqs. (3.3) and (3.4), and by taking $n = 8$, we get the following series solution:

$$u(x) = ax - a^2x^2 + \left(-\frac{4a}{3} + \frac{4a^3}{3}\right)x^3 + \left(2a^2 - 2a^4\right)x^4 + \left(\frac{8a}{15} - \frac{56a^3}{15} + \frac{16a^5}{5}\right)x^5 + \left(-\frac{88a^2}{45} + \frac{328a^4}{45} - \frac{16a^6}{3}\right)x^6 + \left(-\frac{32a}{315} + \frac{1696a^3}{315} - \frac{4544a^5}{315} + \frac{64a^7}{7}\right)x^7 + \left(\frac{344a^2}{315} - \frac{1448a^4}{105} + \frac{1808a^6}{63} - 16a^8\right)x^8 + O(x^9). \tag{3.5}$$

The constant a is evaluated from the boundary conditions given in Eq. (3.2) at $x = 1$ as follows:

$$a = 1.$$

Then, by using the inverse transform rule in Eq. (2.3), we get the following solution:

$$u(x) = x - x^2. \tag{3.6}$$

Note that for $n > 8$ one evaluates the same solution which is the exact solution of Eq. (3.1) with the boundary conditions in Eq. (3.2). This is exactly the same result found by Wazwaz [7].

It is obvious that this is a positive solution in the interval $0 < x < 1$. In Fig. 1, we plot the exact solution given in Eq. (3.6).

Example 3.2. We next consider the following boundary value problem [12]

$$u''(x) + u^2(x) - x^4 - 2 = 0, \quad 0 < x < 1, \tag{3.7}$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 1. \tag{3.8}$$

Taking the differential transform of both sides of Eq. (3.7) and using Theorems 2.1–2.5, the following recurrence relation is obtained:

$$U(k+2) = \frac{1}{(k+1)(k+2)}\left[-\sum_{l=0}^{k} U(l)U(k-l) + \delta(k-4) + 2\delta(k)\right]. \tag{3.9}$$

The boundary conditions in Eq. (3.8) can be transformed as follows:

$$U(0) = 0, \quad U(1) = a. \tag{3.10}$$

Utilizing the recurrence relation in (3.9) and the transformed boundary conditions in Eq. (3.10), the following series solution up to 9-term is obtained:

$$u(x) = ax + x^2 - \frac{a^2}{12}x^4 - \frac{a}{10}x^5 + \frac{a^3}{252}x^7 + \frac{11a^2}{1680}x^8 + \frac{a}{360}x^9 - O(x^{10}), \tag{3.11}$$

where according to Eq. (2.1) $a = u'(0)$.

The constant a is evaluated from the boundary condition given in Eq. (3.8) at $x = 1$ as follows:

$$a = 0.$$

Using the inverse transform rule in Eq. (2.3), we get the following solution:

$$u(x) = x^2. \tag{3.12}$$

For $n > 9$, one evaluates the solution (3.12), which is the exact solution of Eq. (3.7) under the boundary conditions in Eq. (3.8). The result that we obtained is in complete agreement with [12].

It is obvious that (3.12) is a positive solution in $0 < x < 1$. In Fig. 2 we plot the exact solution (3.12).

Example 3.3. We finally consider Troesch's problem [11]

$$u''(x) = \sinh(u(x)), \quad 0 < x < 1, \tag{3.13}$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 1. \tag{3.14}$$

We expand $\sinh u$ around $u_0 = 0$,

$$\sinh u = u + \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \cdots \tag{3.15}$$

Substituting Eq. (3.15) into (3.13), we get

$$u'' = u + \frac{1}{3!}u^3 + \frac{1}{5!}u^5. \tag{3.16}$$

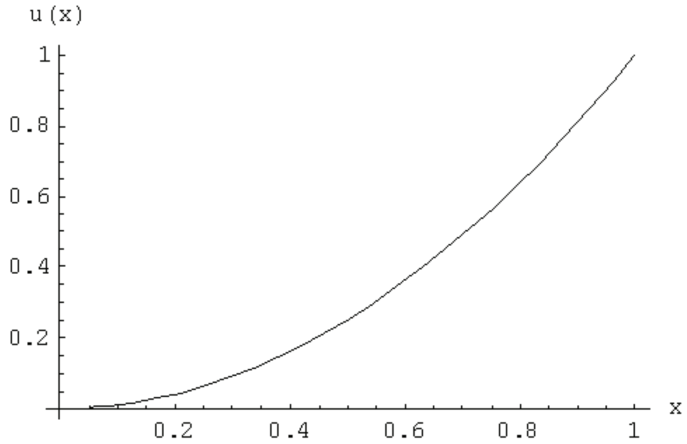

Figure 2: The graph of the exact solution (3.12).

Applying Theorems 2.1, 2.2, 2.3, 2.4 and 2.6 to Eq. (3.16), the following recurrence relation is obtained:

$$U(k+2)=\frac{1}{(k+1)(k+2)}\times\Bigg[U(k)+\frac{1}{3!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U(k_1)U(k_2-k_1)U(k-k_2)$$
$$+\frac{1}{5!}\sum_{k_4=0}^{k}\sum_{k_3=0}^{k_4}\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U(k_1)$$
$$\times U(k_2-k_1)U(k_3-k_2)U(k_4-k_3)U(k-k_4)\Bigg]. \tag{3.17}$$

By using Eqs. (2.1) and (3.14), the boundary conditions at $x_0 = 0$ can be transformed as follows:

$$U(0)=0, \quad U(1)=a. \tag{3.18}$$

Using Eqs. (3.17) and (3.18), and by taking $n = 9$, we get the following series solution:

$$\begin{aligned} u(x) &= ax+\frac{a}{6}x^3+\frac{1}{120}\left(a+a^3\right)x^5+\frac{1}{5040}\left(a+11a^3+a^5\right)x^7 \\ &\quad +\left(\frac{a}{362880}+\frac{17a^3}{60480}+\frac{19a^5}{120960}\right)x^9+O(x^{11}). \end{aligned} \tag{3.19}$$

Imposing the boundary conditions at $x = 1$ on the 9-term approximate solution (3.19) and solving the resulting equation leads to

$$a = 0.84524.$$

Substituting the values of a into (3.19) gives the solution in a series form

$$\begin{aligned} u(x) &= 0.84524x+0.140873x^3+0.0120759x^5 \\ &\quad +0.00157126x^7+0.000239832x^9+O(x^{11}). \end{aligned} \tag{3.20}$$

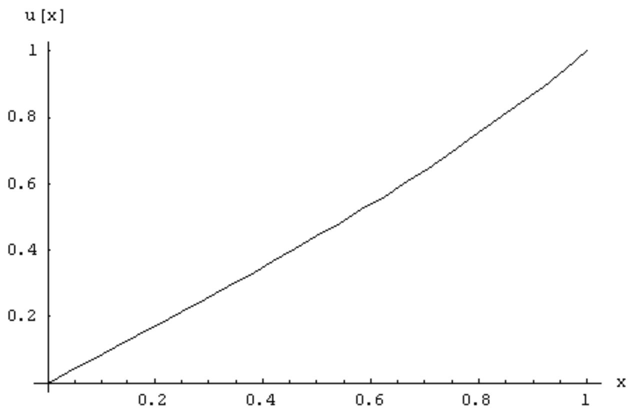

Figure 3: The graph of the approximate solution (3.20).

One can see that (3.20) is a positive solution in $0 < x < 1$. The graph of the approximate solution (3.20) obtained by the present method is shown in Fig. 3.

4. Conclusions

In this study, we have shown that the differential transform method can be succesfully applied for finding the solution of a class of two-point nonlinear boundary value problems. It may be concluded that the differential transform method is an effective and reliable tool in finding the semi numerical-analytic solutions to this type of boundary value problems.

References

[1] R. P. Agarwal. Boundary Value Problems for High Ordinary Differential Equations, World Scientific, Singapore, 1986.

[2] R. P. Agarwal and D. O'Regan. Boundary Value Problems for Superlinear Second Order Ordinary Delay Differential Equations, *J. Differential Equations*, 130:333–335, 1996.

[3] R. P. Agarwal, F. H. Wong, and W. C. Lian. Positive Solutions for Nonlinear Singular Boundary Value Problems, *Appl. Math. Lett.*, 12(2):115–120, 1999.

[4] K. S. Ha and Y. H. Lee. Existence of Multiple Positive Solutions of Singular Boundary Value Problems, *Nonlinear Anal.*, 28:1429–1438, 1997.

[5] D. O'Regan. Theory of Boundary Value Problems, World Scientific, Singapore, 1994.

[6] D. O'Regan. Existence Theory for Ordinary Differential Equations, Kluwer, Boston, MA, 1997.

[7] A. M. Wazwaz. A Reliable Algorithm for Obtaining Positive Solutions for Nonlinear Boundary Value Problems, *Comput. Math. Appl.*, 41:1237–1244, 2001.

[8] M. I. Syam and B. S. Attili. Weighted Residual Method for Obtaining Positive Solutions of Two-Point Nonlinear Boundary Value Problems, *Appl. Math. Comput.*, 176:775–784, 2006.

[9] J. K. Zhou. Differential Transformation and Its Applications for Electrical Circuits, Huazhong University Press, Wuhan, China (in Chinese), 1986.

[10] V. S. Ertürk and S. Momani. Comparing Numerical Methods for Solving Fourth-Order Boundary Value Problems, *Appl. Math. Comput.*, 188(2):1963–1968, 2007.

[11] S. Momani, S. Abuasad, and Z. Odibat. Variational Iteration Method for Solving Nonlinear Boundary Value Problems, *Appl. Math. Comput.*, 183:1351–1358, 2006.

[12] H. Jafari and V. Daftardar-Gejji. Positive Solutions of Nonlinear Fractional Boundary Value Problems Using Adomian Decomposition Method, *Appl. Math. Comput.*, 180:700–706, 2006.