

***A Posteriori* Error Estimates for Schwarz Methods**

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Abstract

A posteriori error estimates for the Schwarz domain decomposition method, for second order boundary value problems, are given. We show that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the interface. After discretisation of the domain by finite elements we adapt the techniques of the residual *a posteriori* error analysis to get an *a posteriori error* estimate for the discrete solutions on subdomains. Numerical results are presented to support the theory.

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1. Introduction

We consider the Schwarz domain decomposition method for the Laplace equation with Dirichlet boundary conditions on the boundary $\partial\Omega$. The *a priori* convergence of the error was given in several papers (see, *e.g.*, Lions [1]), but they do not give a criterion to stop the iterative process. In their paper Otto and Lube [2] gave an *a posteriori* estimate for a non-overlapping domain decomposition algorithm which says that the better the local solutions fit together at the interface the better the errors of the subdomain solution will be. This error estimate enables us to know with certainty when one must stop the iterative process when the required global precision is reached. In the present work, we give an *a posteriori* error estimate in the case of the Schwarz domain decomposition method (*i.e.*, with overlapping) via the introduction of two auxiliary problems defined over two non overlapping subdomains.

The outline of the paper is as follows. In this section, we give a variational formulation of our model problem. We then establish, in Section 2, a stopping criterion for the iterative process in the continuous case. An *a posteriori* error estimate is proposed for

the convergence of the discretised solution using the finite element method on subdomains in Section 3. We finish by an adaptation of the techniques of the residual *a posteriori* error analysis, see Verfürth [3], to give an *a posteriori* estimate in the discrete case. Numerical results are presented to support the theory.

1.1. Problem Formulation and the Domain Decomposition Method

Let us consider the following model problem

$$\begin{cases} -\Delta u = f \in L^2(\Omega), \\ u = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^2 with a sufficiently smooth boundary $\partial\Omega$. The variational formulation of this problem is given by:

$$\begin{cases} \text{Find } u \in V \text{ solution of} \\ a(u, v) = (f, v)_\Omega \quad \forall v \in V, \end{cases} \quad (1.2)$$

where

$$a(u, v) = \int_{\Omega} \nabla u \nabla v \, dx \text{ and } (f, v)_\Omega = \int_{\Omega} f v \, dx \quad \forall v \in V = H_0^1(\Omega).$$

We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12} \text{ and } \partial\Omega_i \cap \Omega_j = \Gamma_i, \quad i \neq j \quad (i, j = 1, 2).$$

The Schwarz alternating method is written as follows: Let u_2^0 be an initial condition on Γ_1 , we construct the sequences $(u_i^{n+1})_{n \in \mathbb{N}}, i = 1, 2$, solutions of

$$\begin{cases} -\Delta u_1^{n+1} = f \text{ on } \Omega_1, \\ u_1^{n+1} = u_2^n \text{ on } \Gamma_1, \\ u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega, \end{cases} \quad (1.3)$$

and

$$\begin{cases} -\Delta u_2^{n+1} = f \text{ on } \Omega_2, \\ u_2^{n+1} = u_1^{n+1} \text{ on } \Gamma_2, \\ u_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega. \end{cases} \quad (1.4)$$

The weak formulations of problems (1.3) and (1.4) are given respectively by:

$$u_1^{n+1} \in V_1 : a_1(u_1^{n+1}, v_1) = (f, v_1) \quad \forall v_1 \in V_1^0; \quad u_1^{n+1} = u_2^n \text{ on } \Gamma_1, \quad (1.5)$$

$$u_2^{n+1} \in V_2 : a_2(u_2^{n+1}, v_2) = (f, v_2) \quad \forall v_2 \in V_2^0, \quad u_2^{n+1} = u_1^{n+1} \text{ on } \Gamma_2, \quad (1.6)$$

where

$$V_i = \{v \in H^1(\Omega_i), v|_{\partial\Omega_i \cap \Omega} = 0\}, \quad V_i^0 = H_0^1(\Omega_i) \text{ and } a_i(u, v) = (\nabla u, \nabla v)_{\Omega_i}.$$

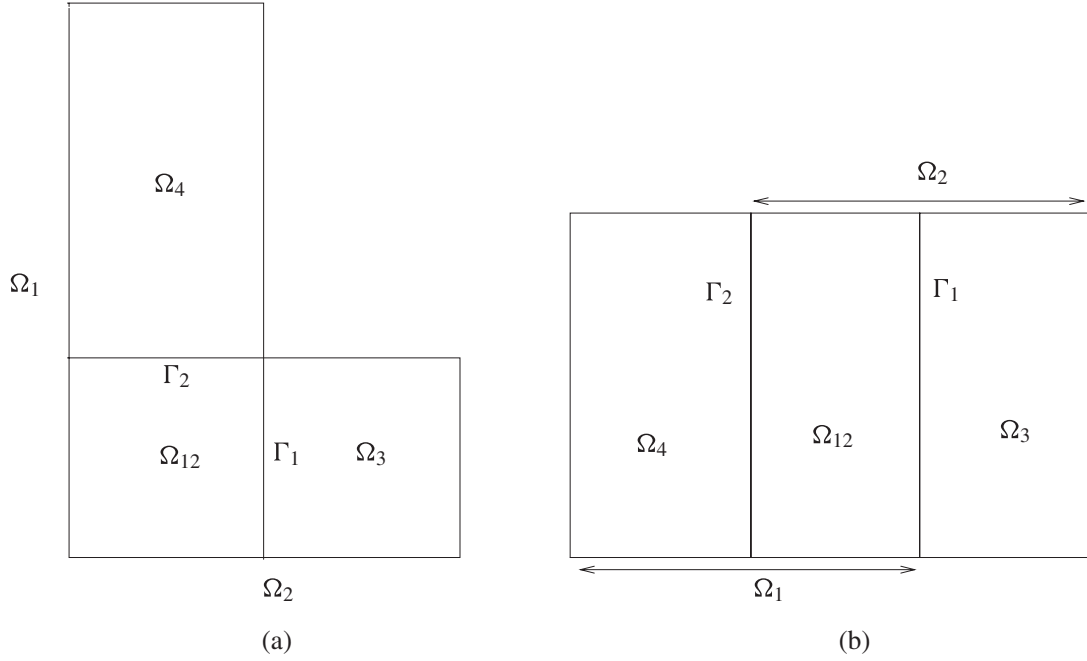


Figure 1: Two examples of domain decompositions (Schur and Schwarz).

2. A Posteriori Error Estimate on Subdomains

To propose such an estimate, we introduce two auxiliary problems defined on two non-overlapping subdomains of Ω and show that the alternating Schwarz method is equivalent to a non-overlapping domain decomposition method. So let us split the domain Ω into two disjoint subdomains: Ω_1 and Ω_3 or Ω_2 and Ω_4 (see Figure 1) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \emptyset \text{ or } \Omega = \Omega_2 \cup \Omega_4 \text{ with } \Omega_2 \cap \Omega_4 = \emptyset.$$

We let (u_1^{n+1}, u_2^{n+1}) be the solutions of problems (1.3) and (1.4). Let us consider the first splitting of Ω : $\Omega = \Omega_1 \cup \Omega_3$ with $\Omega_1 \cap \Omega_3 = \emptyset$ and define the pair $(\bar{u}_1^{n+1}, \bar{u}_2^{n+1})$ by

$$\begin{cases} \bar{u}_1^{n+1} = u_1^{n+1} & \text{on } \bar{\Omega}_1, \\ \bar{u}_2^{n+1} = u_2^{n+1} |_{\bar{\Omega}_3} & \text{on } \bar{\Omega}_3. \end{cases}$$

We have the following result:

Lemma 2.1. The sequences $(\bar{u}_1^{n+1}, \bar{u}_2^{n+1})$ defined previously are solutions of the following problems

$$\begin{cases} -\Delta \bar{u}_1^{n+1} = f & \text{in } \Omega_1, \\ \bar{u}_1^{n+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \frac{\partial \bar{u}_1^{n+1}}{\partial \eta_1} + \bar{u}_1^{n+1} = \frac{\partial \bar{u}_2^n}{\partial \eta_1} + \bar{u}_2^n & \text{on } \Gamma_1, \end{cases} \quad (2.1)$$

$$\begin{cases} -\Delta \bar{u}_2^{n+1} = f & \text{in } \Omega_3, \\ \bar{u}_2^{n+1} = 0 & \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \frac{\partial \bar{u}_2^{n+1}}{\partial \eta_2} = \frac{\partial \bar{u}_1^{n+1}}{\partial \eta_2} & \text{on } \Gamma_1, \end{cases} \quad (2.2)$$

where η_i is the exterior normal to Ω_i .

In a similar manner we can define another auxiliary problem similar to (2.1)–(2.2) over Ω_2 and Ω_4 which is also equivalent to (1.3) and (1.4).

Remark 2.2. We note from Lemma 2.1 that we can go from sequences of solutions to problems (1.5) and (1.6) to sequences of solutions to problems generated by a non overlapping domain decomposition algorithms of Robin type transmission conditions across the interfaces Γ_1 and Γ_2 .

In the following theorem we are going to use this fact to get an *a posteriori* error estimate for the continuous Schwarz method.

Theorem 2.3. Let $u_i = u|_{\Omega_i}$ where u is the solution of problem (1). For the sequences $(u_1^{n+1})_{n \in \mathbb{N}}$, $(u_2^{n+1})_{n \in \mathbb{N}}$ solutions to problems (1.5) and (1.6), we have the following a posteriori error estimation:

$$\left\| u_1^{n+1} - u_1 \right\|_1 + \left\| u_2^n - u_2 \right\|_2 \leq C \left\| u_2^n - u_1^{n-1} \right\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)} \quad (2.3)$$

where C depends on the trace mappings $\text{Tr}_i : V_i \rightarrow L^2(\Gamma_i)$, $i = 1, 2$, and the norm of V_i^0 is the energy norm defined by: $\|u\|_i = \sqrt{a_i(u, u)}$.

Proof. We make use of two auxiliary problems defined each over two non-overlapping subdomains of Ω . The two problems are defined over Ω_1 and Ω_3 and over Ω_2 and Ω_4 respectively. We let

$$\begin{cases} \bar{u}_1^{n+1} = u_1^{n+1} & \text{on } \bar{\Omega}_1, \\ \bar{u}_2^{n+1} = u_2^{n+1}|_{\bar{\Omega}_3} & \text{on } \bar{\Omega}_3. \end{cases}$$

Similarly we let

$$\begin{cases} \hat{u}_1^n = u_1^n|_{\bar{\Omega}_4} & \text{on } \bar{\Omega}_4, \\ \hat{u}_2^n = u_2^n & \text{on } \bar{\Omega}_2. \end{cases}$$

Using Lemma 2.1, the sequences $(\bar{u}_1^{n+1}, \bar{u}_2^{n+1})_{n \in \mathbb{N}}$ are solutions of the non-overlapping domain decomposition problems (2.1) and (2.2) respectively. Similarly for the other two sequences $(\hat{u}_1^n, \hat{u}_2^n)$ defined over Ω_2 and Ω_4 . Since $\bar{u}_1 = u_1$ and $\bar{u}_2 = u_2|_{\Omega_3}$, using [2, Theorem 3.1] we get the following *a posteriori* error estimations:

$$\left\| \bar{u}_1^{n+1} - u_1 \right\|_1 + \left\| \bar{u}_2^n - u_2 \right\|_3 \leq C_1 \left\| \bar{u}_1^{n+1} - \bar{u}_2^n \right\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)}, \quad (2.4)$$

$$\|\widehat{u}_2^n - u_2\|_2 + \|\widehat{u}_1^{n-1} - u_1\|_4 \leq C_2 \|\widehat{u}_2^n - \widehat{u}_1^{n-1}\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}. \quad (2.5)$$

On the other hand we have

$$\begin{aligned} \|u_1^{n+1} - u_1\|_1 + \|u_2^n - u_2\|_2 &\leq \|u_1^{n+1} - \bar{u}_1^{n+1}\|_1 + \|\bar{u}_1^{n+1} - u_1\|_1 \\ &\quad + \|u_2^n - \widehat{u}_2^n\|_2 + \|\widehat{u}_2^n - u_2\|_2 \\ &\leq \|\bar{u}_1^{n+1} - u_1\|_1 + \|\widehat{u}_2^n - u_2\|_2. \end{aligned}$$

Using the inequalities (2.4) and (2.5) we get

$$\|u_1^{n+1} - u_1\|_1 + \|u_2^n - u_2\|_2 \leq C \left\{ \|u_1^{n+1} - u_2^n\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} + \|u_2^n - u_1^{n-1}\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)} \right\}.$$

But from the second equation in (1.3) we have that $u_1^{n+1} = u_2^n$ on Γ_1 . Therefore the first term on the right-hand side of the above inequality vanishes and we get (2.3). \blacksquare

The previous theorem informs us that the error estimate of the solution of the Schwarz alternating method depends on the difference of traces of the subdomain solutions on the interface Γ_2 .

3. A Posteriori Error Estimate in the Discrete Case

In this section, we consider the discretisation of the variational problems (1.5) and (1.6). Let τ_h be a triangulation of Ω compatible with the decomposition and $V_h \subset V$ is the subspace of continuous functions (polynomials of degree k , $k \geq 1$) which vanish over $\partial\Omega$. We have:

$$V_{i,h} = V_h |_{\Omega_i} \quad \text{and} \quad V_{i,h}^0 = \{v_h \in V_{i,h}, v_h |_{\Gamma_i} = 0\}.$$

$W_{i,h} = V_h |_{\Gamma_i}$ is a subspace of $H_{00}^{\frac{1}{2}}(\Gamma_i)$ which consists of continuous piecewise polynomial functions on Γ_i which vanish at the end points of Γ_i ($i = 1, 2$).

Let $u_h \in V_h$ be the solution of the discrete problem associated with (1.2), $u_{i,h} = u_h |_{\Omega_i}$. We construct the sequences $(u_{i,h}^{n+1})_{i=1,2}$, $u_{i,h}^{n+1} \in V_{i,h}$, solutions of the discrete versions of (1.5) and (1.6) at the $(n+1)^{\text{th}}$ iteration, *i.e.*,

$$u_{1,h}^{n+1} \in V_{1,h} : a_1(u_{1,h}^{n+1}, v_{1,h}) = (f, v_{1,h}), \forall v_{1,h} \in V_{1,h}^0, u_{1,h}^{n+1} = u_{2,h}^n \quad \text{on } \Gamma_1, \quad (3.1)$$

$$u_{2,h}^{n+1} \in V_{2,h} : a_2(u_{2,h}^{n+1}, v_{2,h}) = (f, v_{2,h}), \forall v_{2,h} \in V_{2,h}^0, u_{2,h}^{n+1} = u_{1,h}^{n+1} \quad \text{on } \Gamma_2. \quad (3.2)$$

In a similar manner to that of Section 2, we introduce two discrete auxiliary problems, one over Ω_1 and Ω_3 , the other over Ω_2 and Ω_4 . We let

$$\begin{cases} \bar{u}_{1,h}^{n+1} = u_{1,h}^{n+1} \quad \text{on } \bar{\Omega}_1, \\ \bar{u}_{2,h}^{n+1} = u_{2,h}^{n+1} |_{\bar{\Omega}_3} \quad \text{on } \bar{\Omega}_3. \end{cases}$$

Similarly, over Ω_2 and Ω_4 we let

$$\begin{cases} \widehat{u}_{1,h}^n = u_{1,h}^n |_{\overline{\Omega}_4} & \text{on } \overline{\Omega}_4, \\ \widehat{u}_{2,h}^n = u_{2,h}^n & \text{on } \overline{\Omega}_2. \end{cases}$$

We can obtain the discrete counterpart of Theorem 2.3 by doing almost the same analysis as in Section 2 which should be based on a discrete version of Lemma 2.1 (*i.e.*, passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore, the sequences $(\overline{u}_{1,h}^{n+1}, \overline{u}_{2,h}^{n+1})$ and $(\widehat{u}_{1,h}^n, \widehat{u}_{2,h}^n)$ satisfy respectively the discrete counterparts of (10) and (11):

$$\left\| \overline{u}_{1,h}^{n+1} - u_{1,h} \right\|_1 + \left\| \overline{u}_{2,h}^n - u_{2,h} \right\|_3 \leq C_1 \left\| \overline{u}_{1,h}^{n+1} - \overline{u}_{2,h}^n \right\|_{W_{1,h}},$$

$$\left\| \widehat{u}_{2,h}^n - u_{2,h} \right\|_2 + \left\| \widehat{u}_{1,h}^{n-1} - u_{1,h} \right\|_4 \leq C_2 \left\| \widehat{u}_{2,h}^n - \widehat{u}_{1,h}^{n-1} \right\|_{W_{2,h}},$$

where the discrete trace subspace norm $\|\cdot\|_{W_{i,h}}$ defined over Γ_i ($i = 1, 2$) is taken to be

$$\|w_h\|_{W_{i,h}}^2 = \int_{\Gamma_i} w_h^2 dx + \int_{\Gamma_i} \int_{\Gamma_i} \frac{(w_h(x) - w_h(y))^2}{|x - y|^2} dx dy.$$

On the other hand, we have

$$\left\| u_{1,h}^{n+1} - u_{1,h} \right\|_1 + \left\| u_{2,h}^n - u_{2,h} \right\|_2 \leq \left\| \overline{u}_{1,h}^{n+1} - u_{1,h} \right\|_1 + \left\| \widehat{u}_{2,h}^n - u_{2,h} \right\|_2.$$

Hence, using the previous two inequalities, we get

$$\left\| u_{1,h}^{n+1} - u_{1,h} \right\|_1 + \left\| u_{2,h}^n - u_{2,h} \right\|_2 \leq C \left\| u_{2,h}^n - u_{1,h}^{n-1} \right\|_{W_{2,h}}. \quad (3.3)$$

We note that (3.3) is the discrete version of (2.3).

Remark 3.1. Let us note that (3.3) is an estimate of the error between the approximate solution $u_{k,h}$ and the approximate solution of the discretised Schwarz algorithm $u_{k,h}^{n+1}$.

Next we will show an estimate of the error between the approximate solution $u_{i,h}^n$ and the exact solution u . We introduce some necessary notations. We denote by

$$\mathcal{E}_h = \{E \text{ a side of } \tau_h; E \notin \partial\Omega\}.$$

For every $T \in \tau_h$ and $E \in \mathcal{E}_h$, we define

$$\omega_T = \cup \left\{ T' \in \tau_h; T' \cap T \neq \emptyset \right\},$$

and

$$\omega_E = \cup \left\{ T' \in \tau_h; T' \cap E \neq \emptyset \right\}.$$

For a function f which is not necessarily continuous across two neighboring elements of τ_h having E as a common side, $[f]$ denotes the jump of f across E and η_E the normal vector of E .

We have the following theorem which gives the *a posteriori* error estimate.

Theorem 3.2. Let $u_i = u|_{\Omega_i}$ where u is the solution of problem (1), the sequences $(u_{1,h}^{n+1})_{n \in \mathbb{N}}, (u_{2,h}^{n+1})_{n \in \mathbb{N}}$ are solutions of the variational problems (3.1) and (3.2). Then there exists a constant C independent of h such that:

$$\|u_{1,h}^{n+1} - u_1\|_1 + \|u_{2,h}^n - u_2\|_2 \leq C \left\{ \begin{array}{l} \left(\sum_{T \in \tau_h} h_T^2 R_T^2 \right)^{1/2} + \left(\sum_{E \in \varepsilon_h} h_E J_E^2 \right)^{1/2} \\ + \|u_{2,h}^n - u_{1,h}^{n-1}\|_{W_{2,h}} \end{array} \right\} \quad (3.4)$$

where we have denoted

$$R_T = \begin{cases} \|f + \Delta u_{1,h}^{n+1}\|_{0,T} & \text{if } T \subset \overline{\Omega}_1, \\ \|f + \Delta u_{2,h}^n\|_{0,T} & \text{if } T \subset \overline{\Omega}_2, \end{cases}$$

and

$$J_E = \begin{cases} \left\| \left[\frac{\partial u_{1,h}^{n+1}}{\partial \eta_E} \right] \right\|_{0,E} & \text{if } E \subset \Omega_1, \\ \left\| \left[\frac{\partial u_{2,h}^n}{\partial \eta_E} \right] \right\|_{0,E} & \text{if } E \subset \Omega_2, \\ \left\| \frac{\partial u_{1,h}^{n+1}}{\partial \eta_E} \right\|_{0,E} & \text{if } E \subset \Gamma_1, \\ \left\| \frac{\partial u_{2,h}^n}{\partial \eta_E} \right\|_{0,E} & \text{if } E \subset \Gamma_2. \end{cases}$$

Proof. The proof is based on the technique of the residual estimator (see [3]) and Theorem 2.3. We give the main steps.

The symbol $*$ corresponds to $n + 1$ when $i = 1$ and to n when $i = 2$ and we write:

$$\sum_{i=1}^2 \|u_i - u_{i,h}^*\|_i \leq \sum_{i=1}^2 \|u_{i,h} - u_{i,h}^*\|_i + \sum_{i=1}^2 \|u_i - u_{i,h}\|_i. \quad (3.5)$$

The first term on the right-hand side of (3.5) is bounded by (3.3), we are left with the estimation of the second term. We have

$$\sum_{i=1}^2 \|u_i - u_{i,h}\|_i^2 = \sum_{i=1}^2 a_i(u_i - u_{i,h}, u_i - u_{i,h}) \quad (3.6)$$

$$\leq \sum_{i=1}^2 |a_i(u_i - u_{i,h}^*, u_i - v_{i,h})| + \sum_{i=1}^2 \|u_{i,h}^* - u_{i,h}\|_i \|u_i - u_{i,h}\|_i.$$

This last inequality is due to the continuity of the bilinear form $a_i(\cdot, \cdot)$, where $v_{i,h} = v_h|_{\Omega_i}$. By integration by parts we have

$$\begin{aligned} a_i(u_i - u_{i,h}^*, u_i - v_{i,h}) &= \sum_{T \subset \bar{\Omega}_i} \int_T (f + \Delta u_{i,h}^*)(u_i - v_{i,h}) \\ &\quad - \sum_{E \subset \Omega_i} \int_E \left[\frac{\partial u_{i,h}^*}{\partial \eta_E} \right] (u_i - v_{i,h}) ds - \sum_{E \subset \Gamma_i} \int_E \frac{\partial u_{i,h}^*}{\partial \eta_E} (u_i - v_{i,h}) ds. \end{aligned}$$

Hence (3.6) becomes

$$\begin{aligned} \sum_{i=1}^2 \|u_i - u_{i,h}\|_i^2 &\leq \sum_{i=1}^2 \left\{ \begin{aligned} &\sum_{T \subset \Omega_i} \|f + \Delta u_{i,h}^*\|_{0,T} \|u - v_h\|_{0,T} \\ &+ \sum_{E \subset \Omega_i} \left\| \left[\frac{\partial u_{i,h}^*}{\partial \eta_e} \right] \right\|_{0,E} \|u - v_h\|_{0,E} \\ &+ \sum_{E \subset \Gamma_i} \left\| \left[\frac{\partial u_{i,h}^*}{\partial \eta_e} \right] \right\|_{0,E} \|u - v_h\|_{0,E} \end{aligned} \right\} \\ &\quad + \sum_{i=1}^2 \|u_{i,h}^* - u_{i,h}\|_i \|u_i - u_{i,h}\|_i. \end{aligned} \quad (3.7)$$

On the other hand, we use the following standard error estimation results for the interpolation operator $I_h : V \rightarrow V_h$, see [3]:

$$\|w - I_h w\|_{0,T} \leq Ch_T \|w\|_{1,\omega_T} \quad \forall T \in \tau_h \text{ and } \forall w \in V,$$

$$\|w - I_h w\|_{0,E} \leq Ch_E^{\frac{1}{2}} \|w\|_{1,\omega_E} \quad \forall E \in \mathcal{E}_h \text{ and } \forall w \in V,$$

$$\|I_h w\|_T \leq C \|w\|_{1,\omega_T} \quad \forall T \in T_h \text{ and } \forall w \in V.$$

Putting $v_h = u_h + I_h(u - u_h)$ in (3.8) and using the previous three inequalities, we get

$$\begin{aligned} \sum_{i=1}^2 \|u_i - u_{i,h}\|_i^2 &\leq C \left\{ \left(\sum_{T \in \tau_h} h_T^2 R_T^2 \right)^{1/2} + \left(\sum_{E \in \mathcal{E}_h} h_E J_E^2 \right)^{1/2} + \sum_{i=1}^2 \|u_{i,h} - u_{i,h}^*\|_i \right\} \\ &\quad \times \left(\sum_{i=1}^2 \|u_i - u_{i,h}\|_i^2 \right)^{1/2}. \end{aligned}$$

Finally, we substitute this in (3.5), and we use (3.3) to get (3.4). ■

Table 1: The *a posteriori* error estimate Trer on the square.

h	1/8	1/16	1/32	1/64
Er ₁	.4692398(-10)	.5778582(-10)	.8012438(-10)	.1155197(-09)
Er ₂	.3565535(-10)	.4745853(-10)	.9751887(-10)	.1571273(-10)
Trer	.6173353(-09)	.4424475(-09)	.2947041(-09)	.2025559(-09)
Quotient	.1337674	.2378686	.6027851	.946033

3.1. Numerical Examples

As an illustration of the theoretical results obtained in this article and to be able to use the *a posteriori* error estimates, we consider the following problem

$$\begin{cases} -\Delta u = f \in L^2(\Omega), \\ u = 0 \text{ on } \partial\Omega, \end{cases} \quad (3.8)$$

with $\Omega =]0, 1[$ for the first example, $\Omega =]0, 1[\times]0, \frac{1}{2}[\cup]0, \frac{1}{2}[\times]0, 1[$ for the second example (*i.e.*, an *L*-shaped region). The exact solution is taken to be $u(x, y) = xy(x - 1)(y - 1)e^{xy}$ for the first example, $u(x, y) = xy(x - 1)(y - 1)(x - \frac{1}{2})(y - \frac{1}{2})$ for the second example.

We first compute the bilinear Galerkin solution u_h over Ω and then apply the Schwarz method over Ω_1 and Ω_2 to compute the bilinear sequences $u_{i,h}^n$ ($i = 1, 2$) to be able to look at the behavior of the constant C within (15) as the meshsize becomes small ($h^{-1} = 8, 16, 32, 64$). We took $\Omega_1 =]0, \frac{3}{4}[\times]0, 1[$ and $\Omega_2 =]\frac{1}{4}, 1[\times]0, 1[$ for the first example. As for the second example we have $\Omega_1 =]0, 1[\times]0, \frac{1}{2}[$ and $\Omega_2 =]0, \frac{1}{2}[\times]0, 1[$. We denoted by

$$\begin{aligned} \text{Er}_i &= \|u_{i,h} - u_{i,h}^n\|_{1,\Omega_i}, \\ \text{Trer} &= \|u_{2,h}^n - u_{1,h}^{n-1}\|_{W_{2,h}}, \\ \text{Quotient} &= \frac{(\text{Er}_1 + \text{Er}_2)}{\text{Trer}}. \end{aligned}$$

The Schwarz alternating method converges after $n = 6, 7, 9$ and 9 iterations for the first example, after $n = 2, 2, 3$ and 4 iterations for the second example for the mesh sizes ($h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$) respectively. The iterations have been stopped when the absolute error between two subsequent iterates is less than 10^{-8} .

The Quotient in Tables 1 and 2 represents an estimate of the constant C in (3.3). It happens to tend to a constant value around 1 which is what we expect.

Table 2: The *a posteriori* error estimate Trer for the L -shaped region.

h	1/8	1/16	1/32	1/64
Er_1	.2154621(-11)	.6271582(-11)	.1383771(-10)	.2839604(-10)
Er_2	.2574062(-10)	.5159956(-10)	.9962846(-10)	.1940628(-09)
Trer	.2268301(-09)	.2066793(-09)	.8048591(-10)	.3056867(-10)
Quotient	.1229785	.2800045	1.409764	1.4581542

3.2. Concluding Remarks

- 1) In Theorem 3.1 we proved that the subdomain errors depend on the differences of the latter on the interface, on the data f and on the approximate solution u^h . The first two terms of (3.4) are the contribution to the *a posteriori* error from the finite element discretisation whereas the last term is the domain decomposition error. Numerical results indicate that the finite element part of the error estimate dominates the domain decomposition part.
- 2) As far as the asymptotic exactness of the estimator based on (3.4) is concerned, Ainsworth in [4, Theorem 3.5] gave a necessary condition for this property to hold. This condition says that the local residuals in (3.4) should be computed exactly.
- 3) Similar results can be obtained in the general case of more than two subdomains.

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