

Blaschke Structure for a Special Affine Immersion

A. Haji Badali, M. Faghfouri and E. Pourreza

*Faculty of Mathematical Sciences, Tabriz University,
Tabriz, Iran*

E-mail: haji@tabrizu.ac.ir, faghfour@tabrizu.ac.ir, pourreza@tabrizu.ac.ir

Abstract

In this paper we determine a Blaschke structure for affine immersion of Euclidean and hyperbolic type for plane equi-affine curve. In particular, we consider this structure in case that the Ricci tensor of the affine immersion is constant and give a necessary and sufficient condition for The Ricci tensor to be constant.

AMS Subject Classification: 53-XX, 53A15.

Keywords: Affine immersion, Equi-affine curve, Hyperbolic affine immersion, Euclidean affine immersion, Blaschke immersion.

1. Introduction

In this paper, we study nondegenerate affine surfaces in \mathbb{R}^3 . On such surfaces, it is possible to induce an affine connection ∇ , a symmetric bilinear form h which is called the affine metric, and a volume form θ . Details of this construction are considered in Section 2. We then continue with investigating those surfaces for which the immersion is Blaschke. The purpose of this paper is to determine Blaschke structure for affine immersion.

Patrick Lehebel investigated in [1, 2] affine surfaces (and hypersurfaces) which are affine rotation surfaces. In 3-space these surfaces can be characterized by the fact that all affine normals (in the Blaschke sense) intersect a fixed straight line (the axis) and the section with planes containing the axis are shadow boundaries with respect to parallel light. In case when the axis is a proper line (not at infinity) there are three types of surfaces: elliptic ones, hyperbolic ones, and parabolic ones. Friedrich Manhart in [3] investigated the above surfaces with the additional property of vanishing affine Gauss curvature. But yet there is not a classification of affine revolution surfaces whose generator is an equi-affine curve, so in this work we consider those surfaces from this point of view.

This paper consists of five sections. In Section 2, we give some preliminaries which will be needed for our work in Sections 2 and 3. In Section 3, we introduce the concepts of Euclidean and hyperbolic affine immersions with respect to a curve. Then we compute its Ricci tensor component, second fundamental form, and its shape operator.

In Proposition 3.7, by using the concept of affine arc-length we give a necessary and sufficient condition for the Ricci tensor to be constant in terms of the components of the base curve α . In Theorem 3.7, we prove for these curves that the Ricci tensor of immersion f is constant if and only if ζ and η satisfy the following equation $\zeta \eta' = c \neq 0$. Section 4 is devoted to the procedure of producing a transversal and second fundamental form, from a given transversal, in which the related affine immersion into the produced transversal is Blaschke.

2. Preliminaries

In this section we introduce the general notion of affine immersion and some terminology and definitions of affine differential geometry. We shall always assume that the given affine connections have zero torsion.

We now consider two differentiable manifolds with affine connections $(\bar{M}, \bar{\nabla})$ and (M, ∇) of dimensions m and n , respectively. Let $k = m - n$.

Definition 2.1. A differentiable immersion $f : M \rightarrow \bar{M}$ is said to be an *affine immersion* if the following condition is satisfied: There is a k -dimensional differentiable distribution N along $f : x \in M \rightarrow N_x$, a subspace of $T_{f(x)}(\bar{M})$, such that

$$T_{f(x)}(\bar{M}) = f_*(T_x(M)) \oplus N_x, \quad (2.1)$$

$$(\bar{\nabla}_X f_*(Y))_x = (f_*(\nabla_X Y))_x + (\alpha(X, Y))_x, \quad (2.2)$$

at each point $x \in M$ where $X, Y \in \mathfrak{X}(M)$ and $\alpha(X, Y) \in N_x$.

Since the given distribution $x \in M \rightarrow N_x$ is differentiable, each point x has a local basis, namely, a system of k differentiable vectors $\xi_1, \xi_2, \dots, \xi_k$ on a neighborhood U of x that span N_y at each point $y \in U$. This distribution may be regarded as a bundle of transversal k -space. Now we explain the main proposition of the affine immersion.

Proposition 2.2. For a hypersurface immersion $f : M \rightarrow \mathbb{R}^{n+1}$, suppose we have a transversal vector field ξ on M . Then we have a torsion-free induced connection ∇^ξ satisfying

$$D_X f_*(Y) = f_*(\nabla_X^\xi Y) + h^\xi(X, Y)\xi \quad (\text{Gauss}), \quad (2.3)$$

$$D_X \xi = -f_*(S^\xi X) + \tau^\xi(X)\xi \quad (\text{Weingarten}), \quad (2.4)$$

where h^ξ is a symmetric bilinear function on the tangent space $T_x(M)$ and D is the flat connection of \mathbb{R}^3 and S^ξ is a tensor of type $(1, 1)$ called the affine shape operator, and τ^ξ is a 1-form, called the transversal connection form.

Proof. See [4]. ■

For a hypersurface immersion $f : M \rightarrow \mathbb{R}^{n+1}$, a transversal vector field ξ is said to be *equi-affine* if $D_X \xi$ is tangent to M for each $X \in T_x M$, $x \in M$.

Definition 2.3. The *Ricci tensor* of the connection ∇ is defined by

$$\text{Ric}(Y, Z) = \text{trace}\{X \rightarrow R(X, Y)Z\}.$$

Definition 2.4. Let $f : M \rightarrow \mathbb{R}^{n+1}$ be a nondegenerate immersion. If we choose an arbitrary transversal vector field ξ , then we obtain on M the affine fundamental form h , the induced connection ∇ , and the induced volume element θ^ξ . We want to achieve, by an appropriate choice of ξ , the following two goals:

1. (∇, θ^ξ) is an equi-affine structure, that is, $\nabla \theta^\xi = 0$;
2. θ^ξ coincides with the volume element ω_h of the nondegenerate metric h^ξ .

Theorem 2.5. Let $f : M \rightarrow \mathbb{R}^{n+1}$ be a nondegenerate hypersurface immersion. For each point $x_0 \in M$, there is a transversal vector field defined in a neighborhood of x_0 satisfying the conditions (1) and (2) above, such a transversal vector field is unique up to sign.

Proof. See [4]. ■

Definition 2.6. A transversal vector field satisfying (1) and (2) is called *affine normal field* or *Blaschke normal field*. Locally, it is uniquely determined up to sign. For each point $x \in M$ we take the line through x in the direction of the affine normal vector ξ_x . This line, which is independent of the choice of sign for ξ , is called the *affine normal* through x .

3. Euclidean (hyperbolic) Affine Immersion in \mathbb{R}^3

In this section we consider the affine immersion $f : M \rightarrow \mathbb{R}^3$ where M is a two-dimensional affine space. In this case f is called the hypersurface immersion and M is called hypersurface.

Now let $\{u, v\}$ be a flat coordinate system for the affine space $M := \mathbb{R}^2$ with basis $\partial_u := \frac{\partial}{\partial u}$ and $\partial_v := \frac{\partial}{\partial v}$. Let $\xi = (\cos u, \sin u, 0)$ be a transversal vector field for affine immersion $f : M \rightarrow \mathbb{R}^3$ such that

$$f(u, v) = (\zeta(v) \cos u, \zeta(v) \sin u, \eta(v)), \quad \eta' \neq 0.$$

Definition 3.1. With the above notation the affine immersion f is called *Euclidean affine immersion* with $\alpha(v) = (\zeta(v), \eta(v))$ as a generator.

With the same notation, now suppose that

$$\xi = (\cosh u, \sinh u, 0)$$

is a transversal vector field for the affine immersion $f : M \rightarrow \mathbb{R}^3$ with

$$f(u, v) = (\zeta(v) \cosh u, \zeta(v) \sinh u, \eta(v)).$$

Definition 3.2. With the above notation f is called *hyperbolic affine immersion* with $\alpha(v) = (\zeta(v), \eta(v))$ as a generator.

Lemma 3.3. The Ricci tensor of the induced connection (determined by f and ξ) is given by

$$R_{11} = \frac{\zeta' \eta'' - \zeta'' \eta'}{\zeta \eta'} , \quad R_{12} = R_{21} = 0 , \quad R_{22} = 0. \quad (3.1)$$

Proof. From the Gauss equation we have

$$D_{\partial_u} f_*(\partial_u) = f_*(\nabla_{\partial_u}^{\xi} \partial_u) + h(\partial_u, \partial_u)\xi,$$

$$D_{\partial_u} f_*(\partial_v) = f_*(\nabla_{\partial_u}^{\xi} \partial_v) + h(\partial_u, \partial_v)\xi,$$

and

$$D_{\partial_v} f_*(\partial_v) = f_*(\nabla_{\partial_v}^{\xi} \partial_v) + h(\partial_v, \partial_v)\xi.$$

Therefore

$$\begin{aligned} \Gamma_{11}^1 &= 0, & \Gamma_{12}^1 &= \zeta'/\zeta, & \Gamma_{11}^2 &= 0, & \Gamma_{12}^2 &= 0, \\ \Gamma_{21}^1 &= \zeta'/\zeta, & \Gamma_{22}^1 &= 0, & \Gamma_{21}^2 &= 0, & \Gamma_{22}^2 &= \eta''/\eta'. \end{aligned} \quad (3.2)$$

In terms of affine curvature of connection ∇^{ξ} ,

$$R_{ijk}^l = \frac{\partial \Gamma_{lj}^i}{\partial x^k} - \frac{\partial \Gamma_{kj}^i}{\partial x^l} + \Gamma_{lj}^m \Gamma_{km}^i - \Gamma_{kj}^m \Gamma_{lm}^i$$

Now by choosing $x^1 := u$ and $x^2 := v$ we have

$$R_{122}^1 = -R_{121}^2 = (\zeta' \eta'' - \zeta'' \eta')/\zeta \eta',$$

otherwise

$$R_{jkl}^i = 0,$$

therefore from the Ricci equation we can get

$$R_{11} = \frac{\zeta' \eta'' - \zeta'' \eta'}{\zeta \eta'} , \quad R_{12} = R_{21} = 0 , \quad R_{22} = 0$$

that completes the proof of the lemma. ■

In terms of the Gauss formula for the Euclidean immersion we have

$$h^\xi(\partial_u, \partial_u) = \zeta \quad , \quad h^\xi(\partial_v, \partial_v) = (\zeta''\eta' - \zeta'\eta'')/\eta', \quad (3.3)$$

and

$$h^\xi(\partial_u, \partial_v) = h^\xi(\partial_v, \partial_u) = 0.$$

And for the hyperbolic immersion the components of h^ξ are the same as for the Euclidean except $h^\xi(\partial_u, \partial_u) = -\zeta$.

Also by the Weingarten formula we have the components of the shape operator as:

$$S_{11}^\xi = 1/\zeta \quad , \quad S_{22}^\xi = 0, \quad (3.4)$$

and

$$S_{12}^\xi = S_{21}^\xi = 0. \quad (3.5)$$

So we have,

Corollary 3.4. From the above computations the transversal vector field ξ for the Euclidean (hyperbolic) affine immersion is equi-affine.

Definition 3.5. The parameter of a differential curve $\alpha : I \rightarrow \mathbb{R}^2$ is called *affine arc-length* if $|\alpha' \wedge \alpha''| = 1$.

In other words, if $\alpha(t) = (x(t), y(t))$, then the parameter t is affine arc-length if

$$x'y'' - y'x'' = 1.$$

Definition 3.6. The curve α is called *equi-affine curve* if its parameter is arc-length parameter.

Let $\alpha(v) = (\zeta(v), \eta(v))$ be an equi-affine curve. Then we have

Theorem 3.7. Let f be a Euclidean (hyperbolic) affine immersion with equi-affine curve generator α . Then the Ricci tensor of the immersion f is constant if and only if ζ and η satisfy the following equation

$$\zeta\eta' = c \neq 0. \quad (3.6)$$

Proof. Since the parameter of the generator α for the affine immersion f is arc-length, therefore $\zeta'\eta'' - \zeta''\eta' = 1$. From equation (3.1) if $R_{11} = k$ then $\zeta\eta' = 1/k = c$, and *vice versa*. ■

By equation (3.6) and the fact that α has an affine arc-length parameter, we have

$$-c(\zeta')^2 - c\zeta''\zeta = \zeta^2. \quad (3.7)$$

Theorem 3.8. If the Ricci tensor of the Euclidean (hyperbolic) affine immersion is constant, then we have

$$\zeta = \pm\sqrt{\varphi} \quad , \quad \eta = \pm \int k^{-1}\varphi^{-1/2} dv. \quad (3.8)$$

Proof. In equation (3.7) if $\varphi = \zeta^2$, then we have

$$c\varphi'' + 2\varphi = 0.$$

With the same method of Theorem 3.7 and by the assumption $c = 1/k$, and by the solution of the above differential equation we have

$$\begin{aligned} \varphi &= c_1 \cos \sqrt{2k} v + c_2 \sin \sqrt{2k} v, & k > 0, \\ \varphi &= c_1 \cosh \sqrt{-2k} v + c_2 \sinh \sqrt{-2k} v, & k < 0. \end{aligned}$$

It is easy to show that

$$\zeta = \pm\sqrt{\varphi} \quad , \quad \eta = \pm \int k^{-1}\varphi^{-1/2} dv. \quad (3.9)$$

■

4. Blaschke Structure

A well-known choice of relative normals, concerning the induced volume element θ^ξ and the second fundamental form h^ξ , comes from the fact that, there exists a unique choice (up to a sign) of a relative normal ξ which satisfies

$$\theta^\xi = \omega_h$$

where ω_h is the volume element with respect to h^ξ . In this case, one calls ξ the affine normal, h^ξ the affine metric, and f a Blaschke immersion.

In this section we introduce the Blaschke structure for Euclidean and hyperbolic immersion that we have defined in the previous section. As we know from [4], we must take a procedure for finding the affine normal field. We first do this procedure for Euclidean immersion.

Step 1. We must choose a transversal vector field, let $\xi = (\cos u, \sin u, 0)$ be an arbitrary transversal vector field. As we computed in the previous section and from Proposition 3.4 we conclude that $\tau^\xi = 0$.

Step 2. From the computations that we did in the preliminaries section and from equation (3.3) we conclude that the second fundamental form h is nondegenerate.

Step 3. We now determine the induced volume element θ^ξ for ξ that we have chosen in Step 1 by

$$\theta^\xi(\partial_u, \partial_v) = \omega(f_*(\partial_u), f_*(\partial_v), \xi),$$

we have

$$\theta^\xi(\partial_u, \partial_v) = \eta'(v)\zeta(v).$$

Step 4. Now we introduce a unimodular basis $\{X_1, X_2\}$ with

$$\theta^\xi(X_1, X_2) = 1.$$

This condition is a special case of the theorem that we have established for the Ricci tensor in the previous section (Proposition 3.7). So with the choice of this basis we can explain the following proposition.

Proposition 4.1. With the above notation in 4 steps if

$$\eta'(v)\zeta(v) = 1, \tag{4.1}$$

then the basis $\{\partial_u, \partial_v\}$ is unimodular.

Proof. The proof of the proposition is obvious from the definition of unimodular basis. ■

Step 5. As we know by taking $\phi = |\det h|^{1/4}_{\theta^\xi}$ we obtain

$$\phi = \sqrt[4]{\frac{\zeta(\zeta''\eta' - \zeta'\eta'')}{\eta'}}.$$

Now let $\bar{\xi} = \phi\xi + Z$, where Z is to be determined by

$$\tau^\xi + \frac{1}{\phi}h^\xi(Z, \cdot) + d \log \phi = 0.$$

As we know from the previous section $\tau^\xi = 0$, therefore this equation is simply $h(Z, X) = -X\phi$ for every X , so we choose a unimodular bases that we introduced in Step 4 by taking $Z = a\partial_u + b\partial_v$. In this case we have

$$h^\xi(a\partial_u + b\partial_v, X) = -X\phi,$$

first by taking,

$$X = \partial_u,$$

we obtain

$$ah^\xi(\partial_u, \partial_u) = a\zeta = 0,$$

therefore,

$$a = 0.$$

secondly,

$$X = \partial_v,$$

so,

$$\begin{aligned} h^\xi(a\partial_u + b\partial_v, \partial_v) &= -\partial_v\phi, \\ bh^\xi(\partial_v, \partial_v) &= b\frac{\zeta''\eta' - \zeta'\eta''}{\zeta}, \\ b &= -\frac{\zeta}{4}\left(\frac{\zeta}{\eta'}\right)^{-\frac{3}{4}}\left(\frac{\zeta'\eta' - \eta''\zeta}{\eta'^2}\right), \end{aligned}$$

therefore

$$Z = b\partial_v.$$

Step 6. Once we get the affine normal field $\bar{\xi}$, it is easy to compute the affine metric $\bar{h} = h^\xi/\phi$, the affine shape operator S , and the induced connection $\nabla^{\bar{\xi}}$.

Theorem 4.2. Let f be a Euclidean (hyperbolic) affine immersion with equi-affine curve generator α , and ξ, h^ξ be transversal and second fundamental form respectively, then with a choice of ϕ , Z and $\bar{\xi} = \phi\xi + Z$, f is the Blaschke immersion.

Proof. By Steps 1 through 6 and the fact that α is an equi-affine curve we obtain

$$\phi = \sqrt[4]{\frac{\zeta}{\eta'}}.$$

From Step 5 by a simple calculation, we have $Z = b\partial_v$ where

$$b = -\frac{\zeta}{2}\left(\frac{\zeta}{\eta'}\right)^{-3/4}\left(\frac{\zeta'}{\eta'^2}\right).$$

As we saw in Step 6,

$$\bar{\xi} = \sqrt[4]{\frac{\zeta}{\eta'}}\xi + Z.$$

By taking $\bar{\xi}$ as above, f is the Blaschke immersion. ■

Acknowledgments

We would like to thank Prof. F. Manhart and Prof. M. Toomanian for their consistent help and encouragement through many invaluable discussion, suggestions, and stimulating questions during the preparation of this work. Finally, we are grateful to the referee for his/her comments and suggestions that help in improving the manuscript.

References

- [1] P. Lehebel. Affine Rotation Surfaces and Hypersurfaces, Geometry and Topology of Submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995), World Sci. Publ., River Edge, NJ, pp. 198–208, 1996.
- [2] P. Lehebel. Surfaces et Hypersurfaces de Révolution Affine, Thèse de Doctorat de l'Université de Nantes, 1995.
- [3] F. Manhart. Affine Rotational Surfaces with Vanishing Affine Curvature, *J. Geom.*, 80:166–178, 2004.
- [4] K. Nomizu and T. Sasaki. Affine Differential Geometry, Cambridge University Press, 1994.
- [5] M. Do Carmo. Differential Curve and Surface, Prentice Hall, New Jersey, 1976.
- [6] F. J. E. Dillen and L. C. A. Verstraelen. Handbook of Differential Geometry, Vol. 1, North-Holland, 2000.
- [7] R. B. Gardner and G. R. Wilkens. The Fundamental Theorems of Curves and Hypersurfaces in Centro-Affine Geometry, *Bull. Belg. Math. Soc. Simon Stevin*, 4(3):379–401, 1997.
- [8] A. Gray. Modern Differential Geometry of Curves and Surface with Mathematica, CRC Press, 1998.
- [9] H. Guggenheimer. Differential Geometry, McGraw-Hill, New York, 1963.
- [10] S. Kobayashi and K. Nomizu. Foundations of Differential Geometry, Vol. 1 & 2, John Wiley and Sons, New York, 1963 & 1969.
- [11] C. I. Lee and L. Vrancken. Projectively Flat Affine Surfaces with Flat Affine Metric, *J. Geom.*, 70:85–100, 2001.