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Explicit Resolution of Some Singular Barycentric Integral Equations[1]

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Abstract

New singular barycentric integral equations are explicitly solved. The technique used in doing so is that of linear involutionary equations. As an immediate consequence is the resolution of singular integral equations of Cauchy type. Actually, this work is an abstract application of the barycentric transformations.

AMS Subject Classification:
Keywords: Barycentric tranformations, barycentric equations, singular integral equation.

1. Introduction

Actually, the unique source providing the necessary backgrounds regarding the barycentric transformations can be found in [1] and which were partially published in [2]. We shall explicitly solve a new class of singular integral equations which involve a linear finite combination of involutions (barycentric transformations), and so the obtained solution is in turn a combination of barycentric transforms of the second member as well as singular integrals. Regarding the literature on classical singular integral equations we may refer the reader to the works [3–6]. The class of integral equations we are going to study throughout this paper is of the form

$$\begin{aligned} a\sum_{j_1=0}^{p_1-1}\cdots\sum_{j_m=0}^{p_m-1}\frac{\omega_{j_1}\cdots\omega_{j_m}}{\left(\omega_{j_1}-1\right)\cdots\left(\omega_{j_m}-1\right)}\varphi\left(\omega_{j_1}\cdots\omega_{j_m}z\right) \\ +b\sum_{k_1=0}^{q_1-1}\cdots\sum_{k_n=0}^{q_n-1}\frac{\omega_{k_1}\cdots\omega_{k_n}}{\left(\omega_{k_1}-1\right)\cdots\left(\omega_{k_n}-1\right)}\varphi\left(\omega_{k_1}\cdots\omega_{k_n}z\right) \\ +\frac{c}{\pi i}\int_\Gamma\frac{\varphi\left(\xi\right)}{\xi-z}d\xi=\psi\left(z\right) \end{aligned} \tag{1.1}$$

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$(\forall z \in \Gamma)$, where the complex numbers ω_{j_i} are the p_i zeros of the equation

$$z^{p_i} + 1 = 0.$$

Here are the main axes of our study:

- **Notion of barycentric transformations**
    - **Some Examples**
    - **Facts about the Barycentric Transformations**
- **Properties of the singular integral operator**
- **Barycentric Equations**
- **Singular Barycentric Equations**
- **References**

2. Backgrounds

Definition 2.1. A nonempty subset Ω of the complex plane $\mathbb{C}$ is said to be *circular* if $\lambda\Omega = \Omega$ for all $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.

Example 2.2.

- Disks,
- annuli,
- union of annuli,
- union of a disk and annuli,
- the whole plane $\mathbb{C}$.

Definition 2.3. Let Ω be a circular subset of $\mathbb{C}$. A barycentric transformation (at the origin) of order $p \in \mathbb{N}^*$,

$$\mathcal{M}_p : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$$

is defined by

$$\mathcal{M}_p f(z) = \frac{2}{p} \sum_{k=0}^{p-1} \frac{\omega_k}{\omega_k - 1} f(\omega_k z), \ \forall f \in \mathcal{C}(\Omega), \ \forall z \in \Omega, \tag{2.1}$$

where $\omega_0, \omega_1, \ldots, \omega_{p-1}$ are the p zeros of the algebraic equation

$$z^p + 1 = 0, \tag{2.2}$$

explicitly given by

$$\omega_k = \exp\left(\frac{i\pi}{p}(2k+1)\right) \text{ for } k = 0, 1, \ldots, p-1. \tag{2.3}$$

We denote the identity transformation $\mathcal{I}$ by $\mathcal{M}_0$.

Remark 2.4. If $z_0 \in \mathbb{C}$ and $A \subset \mathbb{C}$ such that $A - z_0$ is circular, then the barycentric transformation of order p at z_0 can be defined as follows:

$$\mathcal{M}_{p,z_0} f(z) = \frac{2}{p} \sum_{k=0}^{p-1} \frac{\omega_k}{\omega_k - 1} f(\omega_k (z - z_0) + z_0), \ \forall z \in A. \tag{2.4}$$

Example 2.5.

- For $\Omega = \mathbb{C}$, $f(z) = z^m, m \in \mathbb{N}$, we have for every $p \in \mathbb{N}$, $\mathcal{M}_p f(z) = E_p(m) z^m$, where $E_p(m) = (-1)^{[m/p]}$.
- For $\Omega = \mathbb{C}$, $f(z) = e^z$, we have $\mathcal{M}_2 f(z) = \cos z + \sin z$.
- For $\Omega = \{z \in \mathbb{C} : |z| \neq 1\}$, we have

$$\mathcal{M}_p f(z) = \frac{z^p - 1}{(z^p + 1)(z - 1)}, \quad p \geq 1.$$

We shall denote by $\mathbb{M}(\Omega)$ (Ω being a circular subset of $\mathbb{C}$) the set

$$\{T : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega) : T \text{ is a composition of a finite number of barycentric transformations}\}.$$

We have the following result:

Theorem 2.6. Let Ω be a circular subset of $\mathbb{C}$, then for every $f, g \in \mathcal{C}(\Omega)$ and $R, S, T \in \mathbb{M}(\Omega)$ we have

- $T \circ T = \mathcal{I}$,
- $T(\alpha f + \beta g) = \alpha T f + \beta T g, \ \forall \alpha, \beta \in \mathbb{C}$,
- $S \circ T = T \circ S$,
- $(R \circ S) \circ T = R \circ (S \circ T)$,
- $T\left(z^m f^{(m)}(z)\right) = z^m (Tf)^{(m)}(z), \forall m \in \mathbb{N}, \ \forall z \in \Omega$.

Proof. See [1,2]. ■

It easily follows from the above theorem that every element T of the set $\mathbb{M}(\Omega)$ is an involution.

Proposition 2.7. Let $A = A(r_1, r_2)$ be the open annulus:

$$A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}.$$

If $f : A \to \mathbb{C}$ is given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, then for every $p \geq 1$,

$$\mathcal{M}_{p,z_0} f(z) = \sum_{n=-\infty}^{\infty} a_n E_p(m) (z - z_0)^n, \ \forall z \in A.$$

Proof. See [1,2]. ■

Corollary 2.8. Let $f \in \mathcal{H}(A)$ (the space of all holomorphic functions in A), then for every $p \geq 1$,

$$\frac{1}{p^2} \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \frac{\omega_j \omega_k}{\left(\omega_j - 1\right)(\omega_k - 1)} f\left(z_0 + \omega_j \omega_k (z - z_0)\right) = \frac{1}{4} f(z), \tag{2.5}$$

$\forall z \in A$.

Remark 2.9. Formula (2.5) remains valid for continuous functions! It is in fact a discrete version of the well-known Poincaré-Bertrand commutation formula, namely

$$\frac{1}{(2\pi i)^2} \int_{\Gamma^+} \frac{1}{t - z_0} \int_{\Gamma^+} \frac{f(s)}{s - t} \, ds \, dt = \frac{1}{4} f(z_0), \tag{2.6}$$

where $z_0 \in \Gamma$ (a closed curve in the complex plane), while the integral is defined in the **Cauchy principal value** sense.

We recall the following

Definition 2.10. If Γ is a piecewise Lyapunov curve system and φ is a function defined and integrable on Γ, then the *Cauchy principal value* of the integral, *CPV* for short,

$$\frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\zeta) \, d\zeta}{\zeta - z} \quad (z \in \Gamma), \tag{2.7}$$

is defined as follows

$$\lim_{\epsilon \to 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon} \frac{\varphi(\zeta) \, d\zeta}{\zeta - z} \quad (z \in \Gamma),$$

where $\Gamma_\epsilon = \{\zeta \in \Gamma : |\zeta - z| \geq \epsilon\}$. We shall denote it by S_Γ if it exists.

3. Properties of the Singular Integral Operator S_Γ

The singular integral operator S_Γ satisfies the following properties (see [6]):

- If Γ is a closed Lyapunov curve, then S_Γ is bounded on the space $L^p(\Gamma)$ for $1 < p < \infty$.
- If Γ is a piecewise Lyapunov curve system, then S_Γ is bounded on the space $L^p(\Gamma)$ for $1 < p < \infty$.
- If Γ is an arbitrary piecewise Lyapunov curve system, then S_Γ is an involution on the space $L^p(\Gamma)$ for $1 < p < \infty$.

For instance, if Γ is a piecewise Lyapunov curve system and φ is Hölder continuous of order ν such that $0 < \nu \leq 1$, then the CPV exists at each regular point $z \in \Gamma$.

4. Barycentric Equations

We shall introduce in this section a new class of functional equations that we can explicitly solve. We have the following

Theorem 4.1. Let $a_1, a_2, \ldots, a_m \in \mathbb{C}$ and let $T_1, T_2, \ldots, T_m \in \mathbb{M}(\Omega)$. If $f \in \mathcal{C}(\Omega)$, then the single barycentric equation in the unknown function φ:

$$\sum_{i=1}^{m} a_i T_i \varphi = f, \tag{4.1}$$

is equivalent to an algebraic system of 2^{m-1} equations in 2^{m-1} unknown functions.

Proof. See ([1]). ■

Here are now some applications of the foregoing theorem:

5. Barycentric Singular Integral Equations

We shall state and prove the following lemma which shows that any barycentric transformation commutes with the singular integral operator.

Lemma 5.1. Let Γ be a circle centered at the origin, $T \in \mathbb{M}(\Gamma)$ and let S_Γ be a singular integral operator on $C^\nu(\Gamma)$, the space of Hölder continuous functions of order ν, $0 < \nu \leq 1$, then

$$T \circ S_\Gamma = S_\Gamma \circ T. \tag{5.1}$$

Proof. It suffices to prove that for any $p \in \mathbb{N}$ one has $\mathcal{M}_p S_\Gamma = S_\Gamma \mathcal{M}_p$. Indeed, we have

$$\begin{aligned}
\left(S_\Gamma \mathcal{M}_p\right)(\varphi(z)) &= \int_\Gamma \frac{\mathcal{M}_p(\varphi(\zeta))}{\zeta - z} d\zeta \\
&= \frac{2}{p} \sum_{k=0}^{p-1} \frac{\omega_k}{\omega_k - 1} \int_\Gamma \frac{\varphi(\omega_k \zeta)}{\zeta - z} d\zeta \\
&= \frac{2}{p} \sum_{k=0}^{p-1} \frac{\omega_k}{\omega_k - 1} \int_\Gamma \frac{\varphi(\xi)\,\overline{\omega_k}}{\overline{\omega_k}\xi - z} d\xi \\
&= \frac{2}{p} \sum_{k=0}^{p-1} \frac{\omega_k}{\omega_k - 1} \int_\Gamma \frac{\varphi(\xi)}{\xi - \omega_k z} d\xi \\
&= \left(\mathcal{M}_p S_\Gamma\right)(\varphi(z)),
\end{aligned}$$

for every $z \in \Gamma$ and $\varphi \in C^\nu(\Gamma)$. ■

Here is our main result dealing with the barycentric singular integral equations:

Theorem 5.2. Let Γ be a simple closed curve of $\mathbb{C}$ and let $\widetilde{\Gamma}$ be a circular subset of the complex plane centered at the origin and containing Γ. If $\psi \in C^\nu\left(\widetilde{\Gamma}\right)$ and $T_1, T_2 \in \mathbb{M}\left(\widetilde{\Gamma}\right)$, then the barycentric singular integral equation

$$a_1 T_1 \varphi(z) + a_2 T_2 \varphi(z) + a_3 S_\Gamma \varphi(z) = \psi(z), \tag{5.2}$$

with a_1, a_2 and a_3 in $\mathbb{C}$ such that the corresponding circular determinant

$$\begin{aligned}
\Delta_c &= (a_1 + a_2 + a_3)(a_1 + a_2 - a_3)(a_1 - a_2 - a_3)(a_1 - a_2 + a_3) \\
&= {a_1}^4 + {a_2}^4 + {a_3}^4 - 2(a_1 a_2)^2 - 2(a_2 a_3)^2 - 2(a_1 a_3)^2
\end{aligned} \tag{5.3}$$

is not zero, has a unique solution $\varphi \in C^\nu\left(\widetilde{\Gamma}\right)$ given by

$$\begin{aligned}
\varphi(z) &= \frac{{a_1}^3 - a_1 {a_2}^2 - a_1 {a_3}^2}{\Delta_c} T_1 \psi(z) - \frac{{a_2}^3 - a_2 {a_1}^2 - a_2 {a_3}^2}{\Delta_c} T_2 \psi(z) \\
&+ \frac{{a_3}^3 - a_3 {a_1}^2 - a_3 {a_2}^2}{\Delta_c} S_\Gamma \psi(z) + \frac{2 a_1 a_2 a_3}{\Delta_c} \left(T_1 T_2 S_\Gamma\right) \psi(z),
\end{aligned} \tag{5.4}$$

for all $z \in \Gamma$.

Proof. Without loss of generality we assume that none of T_1 and T_2 is equal to the identity operator. Next, we transform the given equation (5.2) by T_1, T_2, S_Γ and $T_1 T_2 S_\Gamma$ to obtain respectively the following linear system in the unknown variables φ, $T_1 T_2 \varphi$,

$S_\Gamma T_1\varphi,\ S_\Gamma T_2\varphi,$

$$\begin{aligned}
a_1\varphi(z) + a_2 T_1 T_2\varphi(z) + a_3 S_\Gamma T_1\varphi(z) &= T_1\psi(z),\\
a_2\varphi(z) + a_1 T_1 T_2\varphi(z) + a_3 S_\Gamma T_2\varphi(z) &= T_2\psi(z),\\
a_3\varphi(z) + a_1 S_\Gamma T_1\varphi(z) + a_2 S_\Gamma T_2\varphi(z) &= S_\Gamma\psi(z),\\
a_1 S_\Gamma T_2\varphi(z) + a_2 S_\Gamma T_1\varphi(z) + a_3 T_1 T_2\varphi(z) &= T_1 T_2 S_\Gamma\psi(z),
\end{aligned}$$

whose corresponding matrix A_c is given by

$$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_1 & 0 & a_3 \\ a_3 & 0 & a_1 & a_2 \\ 0 & a_3 & a_2 & a_1 \end{pmatrix}. \tag{5.5}$$

We observe that $\det(A_c) = \Delta_c \neq 0$, it follows that the above system admits a unique solution $(\varphi,\ T_1T_2\varphi,\ S_\Gamma T_1\varphi,\ S_\Gamma T_2\varphi)$. Finally, it remains to use classical tools to obtain explicitly the desired solution (5.4). ■

An immediate consequence of the above theorem is the following well known result:

Corollary 5.3. If $T_1 = \mathcal{I}$ and $a_2 = 0$, then equation (5.2) reduces to

$$a_1\varphi(z) + a_3 S_\Gamma\varphi(z) = \psi(z), \tag{5.6}$$

whose unique solution is given by

$$\varphi(z) = \frac{a_1}{\Delta_c}\psi(z) - \frac{a_3}{\Delta_c}\frac{1}{\pi i}\int_\Gamma \frac{\psi(\zeta)\,d\zeta}{\zeta - z}, \tag{5.7}$$

where Δ_c is given by

$$\Delta_c = (a_1 - a_3)(a_1 + a_3) \neq 0.$$

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