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DAD Systems of Control and Observation and Open Problems

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Abstract

The paper considers several kinds of linear hybrid systems described by discrete-difference and difference-differential equations with control. Special attention is paid to the difference-differential hybrid systems in symmetric form. For solutions of such systems, a variation-of-constants formula is proposed and the relative controllability-observability principle is established. For stationary systems, we introduce the determining equations and give solution representations into series along the solutions of their determining equation. Then algebraic properties of the solutions of the determining equation are investigated, in particular, the well-known Hamilton-Cayley matrix theorem is extended to the solutions of the system of determining equations. As a result, parametric criteria for the relative controllability and relative observability are given. Some open problems are formulated.

AMS Subject Classification:
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1. Introduction

In the investigation of real physical processes, one deals both dynamic (differential) and algebraic (functional) dependences. Such processes are described by differential-algebraic systems [1] (some equations in them are differential, and the other are algebraic) or mixed difference-differential systems. One classifies them as hybrid systems [2–5]. However, note that the term “hybrid systems” is overloaded. Nowadays, especially in publications in English, this term is used mainly in connection with discrete-continuous systems or systems containing logical variables [6–8].

In the paper, we consider differential-algebraic delay (DAD) systems to which, in particular, some standard types of linear discrete-continuous and systems with retarded argument of neutral type can be reduced [9]. Such systems can be qualified as hybrid difference-differential systems or quite regular algebraic-differential systems with delay which are, in turn, a special case of descriptor (singular) systems with after-effect. Observe that DAD systems are used in modelling benzene production rectification process [10].

2. Character of Hybrid Systems

In general, hybridity means inhomogeneities in the nature of process to be considered or in the methods to be used in the analysis. The term "hybrid systems" pertains to systems describing the process objects with substantially different characteristics, for example, containing continuous and discrete variables (signals) in the main dynamics, deterministic and random variables or actions, and so on, which eventually specifies the character (nature) of hybrid systems.

There are numerous examples of hybrid systems. In the control field the following pattern of hybrid system is known: linear continuous time independent object described by linear differential equations with discrete linear time independent controller described by finite difference equations, discretely operating recording device being used. These types of systems are commonly studied in the levels called discrete data systems or digital control systems. Another standard example of hybrid control system is switching system where the behaviour can be described by the finite number of dynamical models (systems of differential or difference equations) along with the set of rules for switching between the models. One more field of hybrid systems represents studying qualitative properties (*e.g.*, stability and stabilization, controllability and observability, and so on) of dynamic systems described by difference-differential equations with discontinuous coefficients, systems with variable structure of dynamics. In practice the classic example of hybrid system is heating and cooling systems in dwelling house. A heater and air conditioning along with the characteristics of heat flow form the control system. A thermostat is a discretely randomly controlled system which mainly deals with the symbols "too hot", "too cold", "normal". For more information on hybrid systems, see [2–12].

Nowadays, the problem is that hybrid systems are studied separately, using continuous analysis methods for continuous system dynamics and discrete analysis methods for discrete dynamics, for instance. Below we give an example of unified approach to the construction of mathematical models of dynamical systems described by discrete, difference and differential equations, taking difference-differential hybrid systems as standard systems.

Consider, for example, the simplest discrete continuous hybrid system

$$\begin{cases} \dot{x}(t) &= A_{11}x(t) + A_{12}y[k], \quad t \geq 0, \\ y[k] &= A_{21}x(kh) + A_{22}y[k-1], \quad k = 0, 1, \ldots, \end{cases} \tag{2.1}$$

with initial conditions of the form

$$x(+0) = x(0) = x_0,\ y[-1] = y_0,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^m$, and A_{11}, A_{12}, A_{21}, A_{22} are constant matrices of the corresponding sizes. Denoting

$$\tilde{y}(t) = \begin{bmatrix} x(kh) \\ y[k] \end{bmatrix} \text{ for } t \in [kh, (k+1)h),\ k = 0, 1, \dots,$$

where

$$x(kh) = e^{A_{11}(kh-(k-1)h)}x(kh-h) + \int_{kh-h}^{kh} e^{A_{11}(kh-\tau)}A_{12}y[k-1]\,d\tau,$$

and taking initial conditions as follows

$$x(0) = x_0,\ \tilde{y}(\tau) = \begin{bmatrix} e^{-A_{11}h}\left(x_0 - \int_0^h e^{A_{11}(h-\tau)}A_{12}y_0\,d\tau\right) \\ y_0 \end{bmatrix},\ \ \tau \in [-h, 0),$$

it is not difficult to see [9] that system (2.1) if $y(t) = \tilde{y}(t)$, $t \geq 0$, can be represented as the following difference-differential hybrid system in normal form

$$\begin{cases} \dot{x}(t) &= A_{11}(t)x(t) + A_{12}(t)y(t) + B_1(t)u(t), \\ y(t) &= A_{21}(t)x(t) + A_{22}(t)y(t-h) + B_2(t)u(t), \quad t \geq t_0 \end{cases}$$

with initial conditions

$$x(t_0+0) = x(t_0) = x_0,\ y(\tau) = \psi(\tau),\ \tau \in [t_0 - h, t_0).$$

That is the reason to investigate the hybrid systems, consisting of differential and difference equations.

3. Some Mathematical Models of Hybrid Systems with Delay

Consider descriptor functional-differential system of neutral type

$$\frac{d}{dt}(x(t) - \int_{-h}^{0} d_s G(t,s)x(t+s)) = \int_{-h}^{0} d_s A(t,s)x(t+s) \tag{3.1}$$

$$+ \int_{-h}^{0} d_s B(t,s)u(t+s), \quad t \geq t_0. \tag{3.2}$$

Here $G(t,s)$, $A(t,s)$, $B(t,s)$ are matrices of bounded variation in $[-h,0]$ with respect to s, h is a positive number and $u(\cdot)$ is a control function. If we introduce a new function $y(\cdot)$, we come to system (2) in "hybrid" form

$$\begin{aligned}\dot{y}(t) &= \int_{-h}^{0} d_sA(t,s)x(t+s)+\int_{-h}^{0} d_sB(t,s)u(t+s),\\ x(t) &= y(t)+\int_{-h}^{0} d_sG(t,s)x(t+s), \qquad t\geq t_0.\end{aligned}$$

If the measure in (3.2) is discrete and concentrated at the points $s=-h_j$, $j=0,1,\dots,l$; $0=h_0<h_1<\cdots<h_l=h$ and $d_sG(t,s)\to 0$ as $s\to 0$, then we obtain DAD system with concentrated delays

$$\dot{x}(t) = \sum_{j=0}^{l}\left(A_{11j}(t)x(t-h_j)+A_{12j}(t)y(t-h_j)+B_{1j}(t)u(t-h_j)\right), \tag{3.3}$$

$$y(t) = \sum_{j=0}^{l}\left(A_{21j}(t)x(t-h_j)+A_{22j}(t)y(t-h_j)+B_{2j}(t)u(t-h_j)\right), \quad t\geq 0, \tag{3.4}$$

with initial conditions of the form

$$\begin{gathered}x(t_0+0)=x(t_0)=x_0\in\mathbb{R}^n,\ x(\tau)=\varphi(\tau),\\ y(\tau)=\psi(\tau),\ u(\tau)=\xi(\tau),\ \tau\in[t_0-h_1,t_0).\end{gathered} \tag{3.5}$$

Here $A_{11j}(t)\in\mathbb{R}^{n\times n}$, $A_{12j}(t)\in\mathbb{R}^{n\times m}$, $A_{21j}(t)\in\mathbb{R}^{m\times n}$, $A_{22j}(t)\in\mathbb{R}^{m\times m}$, $B_{1j}(t)\in\mathbb{R}^{n\times r}$, $B_{2j}(t)\in\mathbb{R}^{m\times r}$; $j=0,1,\dots,l$; $A_{220}=0$, and the components of the vector functions $\psi(t)$, $\varphi(t)$, $\xi(t)$, $t\in[t_0-h_1,t_0]$ and the control $u(\cdot)$ being piecewise continuous. The corresponding solution of system (3.3)–(3.5) will be denoted by the symbols

$$\begin{aligned}x(t) &= x(t,t_0,x_0,\varphi,\psi,\xi,u),\\ y(t) &= y(t,t_0,x_0,\varphi,\psi,\xi,u), \quad t\geq t_0.\end{aligned}$$

Notice that system (3.3)–(3.5) is a generalization of neutral type concentrated delay. On the other side, it is a particular case of descriptor time delay system (2.1). The simplest time invariant DAD system is the following one

$$\begin{cases}\dot{x}(t) &= A_{11}x(t)+A_{12}y(t)+B_1u(t),\\ y(t) &= A_x(t)+Ay(t-h)+Bu(t), \quad t\geq 0.\end{cases} \tag{3.6}$$

4. Available Results

Taking into account 2-D system grounds, we pay attention (for more general results see [13]) to the simplest DAD system in symmetric form with respect to the differential and shift operators

$$\begin{cases} \dot{x}(t) &= A_{11}(t)x(t) + A_{12}(t)y(t) + B_1(t)u(t), \\ y(t+h) &= A_{21}(t)x(t) + A_{22}(t)y(t) + B_2(t)u(t), \quad t \geq t_0, \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, $y(t+h) \in \mathbb{R}^m$, $t \geq t_0$; the entries of the matrix-valued functions $A_{11}(t) \in \mathbb{R}^{n\times n}$, $A_{12}(t) \in \mathbb{R}^{n\times m}$, $A_{21}(t) \in \mathbb{R}^{m\times n}$, $A_{22}(t) \in \mathbb{R}^{m\times m}$, $B_1(t) \in \mathbb{R}^{n\times r}$, $B_2(t) \in \mathbb{R}^{m\times r}$ are continuous.

For system (4.1), we consider the following initial-value problem

$$x(t_0+0) = x_0, \quad y(\tau) = \psi(\tau), \quad \tau \in [t_0, t_0+h), \tag{4.2}$$

where $x_0 \in \mathbb{R}^n$ and ψ is a piecewise continuous m-vector function in $[t_0, t_0+h]$.

A solution $x(t; t_0, x_0, \psi, u)$, $y(t+h; t_0, x_0, \psi, u)$ for $t \geq t_0$ is defined as follows: $x(\cdot)$ is a continuous n-vector function and $y(\cdot)$ is piecewise continuous m-vector function, satisfying the first equation (4.1) for $t \geq t_0$ except at the points $t \neq t_0+kh$, $k = 0, 1, \ldots,$ and the second one for $t \geq t_0$.

For ψ and u are differentiable, system (4.1) can be reduced to a neutral type time-delay equation and, as a result, the variation-of-constants formula can be used to represent the system solutions. In the general case, the situation is more complicated. However, by using the method of steps, one can state that the solution of system (4.1) corresponding to initial problem (4.2) and to a piecewise continuous control exists and is unique.

Let us denote $T_t = \left[\dfrac{t-t_0}{h}\right]$ and define matrix functions $X^*(t,\tau)$, $Z^*(t,\tau)$, $Y^*(t,\tau)$ as solutions of a dual system of the form:

$$-\frac{\partial X^*(t,\tau)}{\partial \tau} = X^*(t,\tau)A_{11}(\tau) + Y^*(t,\tau)A_{21}(\tau),$$

$$\tau \leq T,\ \tau \neq t-kh,\ k = 1, 2, \ldots, T_t;$$

$$Y^*(t,\tau-h) = X^*(t,\tau)A_{12}(\tau) + Y^*(t,\tau)A_{22}(\tau), \quad \tau \leq t;$$

$$Y^*(t,\tau) = 0, \quad \tau > t;$$

$$X^*(t,t-kh-0) - X^*(t,t-kh+0) = Z^*(t,t-kh)A_{21}(t-kh), \quad k = 1, 2, \ldots, T_t;$$

$$Z^*(t,t-kh-h) = Z^*(t,t-kh)A_{22}(t-kh), \quad k = 1, 2, \ldots, T_t - 1.$$

According to the dual system the solutions $X^*(t,\tau)$ and $Y^*(t,\tau)$ are piecewise continuous with respect to argument τ matrix functions with jumps at the points $\tau = t-kh$, $k = 1, 2, \ldots, T_t$. The matrix function $Z^*(t, t-kh)$ is regarded as discrete with respect to the second argument. The initial and boundary conditions for the dual system under consideration are established by the last three equations.

Theorem 4.1. The solution of system (4.1) with initial conditions (4.2) and piecewise continuous control $u(t), t > t_0$, exists, is unique and can be written as follows (variation-of-constants formula):

$$
\begin{aligned}
&X^*(t, t_0+0)x_0 + \int_{t_0}^{t_0+h} Y^*(t, \tau-h)\psi(\tau)\, d\tau \\
&+ \int_{t_0}^{t} (X^*(t,\tau)B_1(\tau) + Y^*(t,\tau)B_2(\tau))u(\tau)\, d\tau \\
&+ \sum_{k=1}^{T_t} Z^*(t, t-kh)B_2(t-kh)u(t-kh) + Z^*(t, t-T_t h)\psi(t-T_t h) \\
&= \left\{ \begin{array}{ll} x(t) \text{ for } t \geq t_0 & \text{if } X^*(t, t-0) = I_n \in \mathbb{R}^{n\times n} \\ & \text{and } Z^*(t, t-h) = 0 \in \mathbb{R}^{n\times m}; \\ y(t) \text{ for } t \geq t_0 + h & \text{if } X^*(t, t-0) = 0 \in \mathbb{R}^{m\times n} \\ & \text{and } Z^*(t, t-h) = I_m \in \mathbb{R}^{m\times m}; \end{array} \right\}.
\end{aligned}
$$

For the stationary system

$$
\begin{aligned}
&A_{11}(t) = A_{11},\ A_{12}(t) = A_{12},\ A_{21}(t) = A_{21}, \\
&A_{22}(t) = A_{22},\ B_1(t) = B_1,\ B_2(t) = B_2,\ t_0 = 0.
\end{aligned} \tag{4.3}
$$

Theorem 4.1 can be detailed if we take into account that in such a case

$$X^*(t,\tau) = X(t-\tau),\ Y^*(t,\tau) = Y(t-\tau),\ Z^*(t,\tau) = Z(t-\tau)$$

and the dual system can be rewritten as

$$\dot{X}(t) = X(t)A_{11} + Y(t)A_{21},\ t \geq 0,\ t \neq kh,\ k = 1, \dots, T_t;$$

$$Y(t+h) = X(t)A_{12} + Y(t)A_{22},\ t \geq 0;$$

$$Y(t) = 0,\ t < h;$$

$$X(kh) - X(kh-0) = Z(kh)A_{21}, \quad k = 1, 2, \dots, T_t; \tag{4.4}$$

$$Z(kh+h) = Z(kh)A_{22}, \quad k = 1, 2, \dots, T_t - 1. \tag{4.5}$$

Remark 4.2. If $T_t = 0$, then equations (4.4) and (4.5) are absent, equation (4.5) disappears also in the case of $T_t = 1$.

Theorem 4.3. The solution of system (4.1)–(4.3) with $t_0 = 0$ and a piecewise continuous control $u(t), t > 0$, exists, is unique and can be written as follows

$$
\begin{aligned}
x(t, t_0, x_0, \psi, u) = X(t)x_0 &+ \int_0^h Y(t-\tau+h)\psi(\tau)\, d\tau \\
&+ \int_0^h (X(t-\tau)B_1 + Y(t-\tau)B_2)u(\tau)\, d\tau
\end{aligned} \tag{4.6}
$$

$$\text{for } t \geq 0 \text{ if } X(0) = I_n \in \mathbb{R}^{n\times n},\ Z(h) = 0 \in \mathbb{R}^{n\times m};$$

$$\begin{aligned} y(t, t_0, x_0, \psi, u) = X(t-0)x_0 &+ \int_0^h Y(t-\tau+h)\psi(\tau)\,d\tau \\ &+ \int_0^h (X(t-\tau)B_1 + Y(t-\tau)B_2)u(\tau)\,d\tau \\ &+ \sum_{k=1}^{T_t} Z(kh)B_2(t-kh) + Z(T_t h)\psi(t-T_t h) \end{aligned} \tag{4.7}$$

$$\text{for } t \geq h \text{ if } X(0) = 0 \in \mathbb{R}^{m\times n},\ Z(h) = I_m \in \mathbb{R}^{m\times m}.$$

Introduce a determining equation of system (3.2)–(3.4):

$$\begin{aligned} X_{k+1}(t) &= A_{11}X_k(t) + A_{12}Y_k(t) + B_1U_k(t), \\ Y_k(t+h) &= A_{21}X_k(t) + A_{22}Y_k(t) + B_2U_k(t), \end{aligned} \tag{4.8}$$

$$k = 0, 1, \ldots; \quad t \geq 0,$$

with initial conditions

$$X_k(t) = 0,\ Y_k(t) = 0 \text{ if } k < 0 \text{ or } t < 0;$$

$$U_0(0) = I_r,\ U_k(t) = 0 \text{ if } k^2 + t^2 \neq 0.$$

In the sequel, we give some algebraic properties of the solutions of the determining equation.

By induction, we can prove Lemmas 4.4–4.6.

Lemma 4.4. The following identities are valid:

$$\begin{aligned} &\big(A_{11} + A_{12}\omega(I_m - A_{22}\omega)^{-1}A_{21}\big)^{k-1} \\ &\quad\times \big(B_1 + A_{12}\omega(I_m - A_{22}\omega)^{-1}B_2\big) \equiv \sum_{j=0}^{+\infty} X_k(jh)\omega^j, \end{aligned}$$

$$\begin{aligned} &(I_m - A_{22}\omega)^{-1}A_{21}\omega\big(A_{11} + A_{12}\omega(I_m - A_{22}\omega)^{-1}A_{21}\big)^{k-1} \\ &\quad\times \big(B_1 + A_{12}\omega(I_m - A_{22}\omega)^{-1}B_2\big) \equiv \sum_{j=0}^{+\infty} Y_k(jh)\omega^j, \end{aligned}$$

$$k = 1, 2, \ldots,$$

$$(I_m - A_{22}\omega)^{-1}B_2\omega \equiv \sum_{j=0}^{+\infty} Y_0(jh)\omega^j,$$

where $|\omega| < \omega_1$ and ω_1 is a sufficiently small positive number.

Lemma 4.5. The following identities are valid:

$$\begin{aligned}
&\big(A_{22} + A_{21}\omega(I_n - A_{11}\omega)^{-1}A_{12}\big)^{j-1} \\
&\quad\times \big(B_2 + A_{21}\omega(I_n - A_{11}\omega)^{-1}B_1\big) \equiv \sum_{k=0}^{+\infty} Y_k(jh)\omega^k, \\
&(I_n - A_{11}\omega)^{-1}A_{12}\omega\big(A_{22} + A_{21}\omega(I_n - A_{11}\omega)^{-1}A_{12}\big)^{j-1} \\
&\quad\times \big(B_2 + A_{21}\omega(I_n - A_{11}\omega)^{-1}B_1\big) \equiv \sum_{k=0}^{+\infty} X_k(jh)\omega^k, \\
&j = 1, 2, \ldots; \\
&(I_n - A_{11}\omega)^{-1}B_1\omega \equiv \sum_{k=0}^{+\infty} X_k(0)\omega^k,
\end{aligned}$$

$|\omega| < \omega_1$, ω_1 is a sufficiently small positive number.

Let us introduce the notation:

$$\alpha(\omega) = \det(I_m - A_{22}\omega) \in \mathbb{R}^1[\omega],$$

$$\beta(\omega) = \det(I_n - A_{11}\omega) \in \mathbb{R}^1[\omega].$$

$Q_1^*(\omega) \in \mathbb{R}^{m\times m}[\omega]$ and $Q_2^*(\omega) \in \mathbb{R}^{n\times n}[\omega]$ are the adjoint of the matrices $(I_m - A_{22}\omega)$ and $(I_n - A_{11}\omega)$ respectively.

Introduce the characteristic equation of the matrix $\big(A_{11} + A_{12}\omega(I_m - A_{22}\omega)^{-1}A_{21}\big)$:

$$\begin{aligned}
0 = \Delta(\lambda) &= \det(\lambda I_n - \big(A_{11} - A_{12}\omega(I_m - A_{22}\omega)^{-1}A_{21}\big) \\
&= \det\left(\lambda I_n - A_{11} - \frac{A_{12}Q_1^*A_{21}}{\alpha(\omega)}\right) \\
&= \frac{1}{(\alpha(\omega))^n}\det(\lambda\alpha(\omega)I_n - A_{11}\alpha(\omega) - A_{12}\omega Q_1^*(\omega)A_{21}).
\end{aligned} \tag{4.9}$$

Consider the characteristic equation of the matrix $\big(A_{22} + A_{21}\omega(I_n - A_{11}\omega)^{-1}A_{12}\big)$:

$$\begin{aligned}
0 = \Delta(\lambda) &= \det(\lambda I_m - \big(A_{22} - A_{21}\omega(I_n - A_{11}\omega)^{-1}A_{12}\big) \\
&= \det\left(\lambda I_m - A_{22} - \frac{A_{21}\omega Q_2^*A_{12}}{\beta(\omega)}\right) \\
&= \frac{1}{(\beta(\omega))^m}\det(\lambda\beta(\omega)I_m - A_{22}\beta(\omega) - A_{21}\omega Q_2^*(\omega)A_{12}).
\end{aligned} \tag{4.10}$$

Suppose $|\omega| < \omega_1$, where ω_1 is a sufficiently small positive number. Then (4.9) and (4.10) can be represented as

$$\sum_{i=0}^{n}\sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^{j} = 0 \quad \text{and} \quad \sum_{i=0}^{m}\sum_{j=0}^{nm} p_{ij}\lambda^{m-i}\omega^{j} = 0$$

respectively.

Here r_{ij} and p_{kj}, $i = 0, 1, \ldots, n$; $k = 0, 1, \ldots, m$; $j = 0, 1, \ldots, nm$ are real numbers with $r_{00} = 1$ and $p_{00} = 1$.

The last two equations can be rewritten as

$$\lambda^{n} = -\sum_{j=1}^{nm} r_{0j}\lambda^{n}\omega^{j} - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^{j},$$

$$\lambda^{m} = -\sum_{j=1}^{nm} p_{0j}\lambda^{m}\omega^{j} - \sum_{i=1}^{m}\sum_{j=0}^{nm} p_{ij}\lambda^{m-i}\omega^{j}, \ |\omega| < \omega_1.$$

Lemma 4.6. The solutions of the determining equations (4.8) satisfy the following conditions:

$$\begin{aligned}
X_\gamma(kh) = &-\sum_{j=1}^{\min\{k,nm\}} r_{0j}X_\gamma((k-j)h) \\
&-\sum_{i=1}^{n}\sum_{j=0}^{\min\{k,nm\}} r_{ij}X_{\gamma-i}((k-j)h), \\
Y_\gamma(kh) = &-\sum_{j=1}^{\min\{k,nm\}} r_{0j}Y_\gamma((k-j)h) \\
&-\sum_{i=1}^{n}\sum_{j=0}^{\min\{k,nm\}} r_{ij}Y_{\gamma-i}((k-j)h), \\
X_k(\nu h) = &-\sum_{j=1}^{\min\{k,nm\}} p_{0j}X_{k-j}(\nu h) \\
&-\sum_{i=1}^{m}\sum_{j=0}^{\min\{k,nm\}} p_{ij}X_{k-j}((\nu-i)h),
\end{aligned}$$

$$
\begin{aligned}
Y_k(\nu h) = &- \sum_{j=1}^{\min\{k,nm\}} p_{0j} Y_{k-j}(\nu h) \\
&- \sum_{i=1}^{m} \sum_{j=0}^{\min\{k,nm\}} p_{ij} Y_{k-j}((\nu - i)h),
\end{aligned}
$$

for $k = 0, 1, \ldots$, $\nu = m + 1, \ldots$, and $\gamma = n + 1, \ldots$, where $\sum_{k=i}^{j}(\cdots) = 0$ if $j < i$.

Lemma 4.6 for $\gamma = n + 1$ and for $\nu = m + 1$ can be regarded as generalizations of the well-known theorem of Hamilton-Cayley from matrix theory to solutions of the determining equations (4.8).

To apply the Laplace transform we need

Theorem 4.7. Suppose $\max_{t\in[0,h]} \|\psi(t)\| = M_1$ and $\|u(t)\| \leq M_2 e^{\sigma t}$, $t \geq 0$ (where M_1, M_2 and σ are the positive constants), then there exist positive numbers N and α such that all solutions of the system (4.1)–(4.3) satisfy the following conditions

$$
\|x(t)\| \leq N e^{\alpha t}, \quad \|y(t)\| \leq N e^{\alpha t}, \quad t \geq 0,
$$

where N, α are defined only by M_1, M_2, σ and the system parameters.

Applying the Laplace transform to system (4.1)–(4.3), we obtain the formulae for the representation of solutions of the system into series of their determining equation solutions.

Theorem 4.8. The solution of the system (4.1)–(4.3) with piecewise continuous control $u(\tau)$, $\tau \geq 0$, exists, is unique and can be represented by the following formulae:

$$
\begin{aligned}
x(t, 0, x_0, \psi, u) = &\sum_{k=0}^{+\infty} \sum_{i=1}^{[t/h]} X_{k+1}(ih) \int_0^{t-ih} \frac{(t - \tau - ih)^k}{k!} u(\tau)\, d\tau \\
&+ x(t, 0, x_0, \psi, 0), \quad t > 0,
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
y(t, 0, x_0, \psi, u) = &\sum_{k=0}^{+\infty} \sum_{i=1}^{[t/h]} Y_{k+1}(ih) \int_0^{t-ih} \frac{(t - \tau - ih)^k}{k!} u(\tau)\, d\tau \\
&+ \sum_{i=1}^{[t/h]} Y_0(ih) u(t - ih) + y(t, 0, x_0, \psi, 0), \quad t \geq 0,
\end{aligned} \tag{4.12}
$$

where the vector functions $x(t, 0, x_0, \psi, 0)$, $y(t, 0, x_0, \psi, 0)$ depend on the initial data only.

The results formulated can be applied to the investigation of such basic problems of the qualitative control and observation theory of DAD systems as controllability, observability and duality.

Definition 4.9. System (4.1), (4.2) is called:

i) $\mathbb{R}^n$-*controllable with respect to* x *on* $[t_0, t_*]$ if for any x_0, $x_* \in \mathbb{R}^n$ and for any piecewise continuous m-vector function $\psi(\tau)$, $\tau \in [t_0, t_0 + h]$, there exists a piecewise continuous control $u(\tau)$, $\tau \in [t_0, t_*]$, such that

$$x(t_*, t_0, x_0, \psi, u) = x_*; \tag{4.13}$$

ii) $\mathbb{R}^n$-*attainable with respect to* x *on* $[t_0, t_*]$ if (19) holds for $x_0 = 0$ and $\psi(\tau) = 0$, $\tau \in [t_0, t_0 + h]$.

Similarly, the notion of $\mathbb{R}^n$-*controllability with respect to* y is defined.

Consider the following dual system ($t_* > t_0 + h$):

$$\begin{aligned} &\frac{d}{d\tau} x^*(\tau) + x^*(\tau)A_{11}(\tau) + y^*(\tau)A_{21}(\tau) = 0, \\ &y^*(\tau - h) - x^*(\tau)A_{12}(\tau) - y^*(\tau)A_{22}(\tau) = 0, \quad \tau \in [t_0, t_*], \end{aligned} \tag{4.14}$$

with initial conditions

$$x^*(t_* - 0) = x^*(t_*) = x^*, \;\; y^*(\tau) = \psi^*(\tau), \;\; \tau \in [t_* - h, t_*],$$

and the output

$$\begin{aligned} z^*(t) &= z^*(t, t_*, x^*, \psi^*) \\ &= x^*(\tau)B_1(\tau) + y^*(\tau)B_2(\tau), \;\; \tau \in [t_0, t_*], \end{aligned} \tag{4.15}$$

where $x^*(\tau) \in \mathbb{R}^{1\times n}$, $y^*(\tau) \in \mathbb{R}^{1\times m}$ and ψ^* is a piecewise continuous m-row function.

Definition 4.10. System (4.14), (4.15) is said to be $\mathbb{R}^n$-*observable with respect to* x *on* $[t_0, t_*]$ if

$$z^*(t, t_*, x^*, \psi^*) = z^*(t, t_*, p^*, \psi^*) \text{ a.e. } t \in [t_0, t_*] \Rightarrow x^* = p^*.$$

Lemma 4.11. Along the trajectories of the basic (4.1) and the dual (4.14), (4.15) systems, the following duality correlation is valid:

$$\begin{aligned} x^*(t_* - 0)x(t_* - 0) = x^*(t_0 + 0)x(t_0 + 0) & \\ + \int_{t_0}^{t_0+h} y^*(t - h)\psi(t)\,dt &- \int_{t_*}^{t_*+h} y^*(t - h)y(t)\,dt \\ + \int_{t_0}^{t_*} z^*(t)u(t)\,dt. & \end{aligned}$$

Theorem 4.12. The following statements are equivalent:

i) System (4.1) is $\mathbb{R}^n$-controllable with respect to x on $[t_0, t_*]$;

ii) System (4.1) is $\mathbb{R}^n$-attainable with respect to x on $[t_0, t_*]$;

iii) The matrix rows

$$X^*(t_*, \tau)B_1(\tau) + Y^*(t_*, \tau)B_2(\tau),\ \ \tau \in [t_0, t_*],$$

are linearly independent a.e. in $[t_0, t_*]$;

iv) System (4.14), (4.15) is $\mathbb{R}^n$-observable with respect to x on $[t_0, t_*]$.

Along with system (4.1)–(4.3), consider the following dual system

$$\begin{cases} \dot{x}^*(t) & = x^*(t)A_{11} + y^*(t)A_{21}, \\ y^*(t+h) & = x^*A_{12} + y^*(t)A_{22}, \end{cases} \tag{4.16}$$

$$x(+0) = x(0) = x_0^*,\ \ y(\tau) = \psi^*(\tau),\ \ \tau \in [-h, 0),$$

with the output

$$z(t) = z(t; x_0^*, \psi^*) = x^*(t)B_1 + y^*(t)B_2 \tag{4.17}$$

Definition 4.13. System (4.1)–(4.3) is called t_1-*controllable* for $t_1 > h$ if for any vector $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in \mathbb{R}^{n+m}$ and for any initial conditions (4.2) there exists a piecewise continuous control $u(\cdot)$ such that the condition $\begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ holds for the corresponding solution $x(t)$, $y(t)$ of the system.

Then we can formulate the following parametric controllability and observability criteria for time stationary systems.

Theorem 4.14. System (4.1)–(4.3) is relatively t_1-controllable if and only if

$$\operatorname{rank}\left[\begin{bmatrix} X_k(ih) \\ Y_k(ih) \end{bmatrix},\ k = 0, 1, \ldots, n;\ i = 0, 1, \ldots, \min\{T_{t_1-0}, m\}\right] = m + n.$$

Theorem 4.15. System (4.1)–(4.3) is relatively t_1-controllable with respect to x if and only if

$$\operatorname{rank}\left[X_k(ih),\ k = 1, 2, \ldots, n;\ i = 0, 1, \ldots, \min\{T_{t_1-0}, m\}\right] = n.$$

Theorem 4.16. System (4.1)–(4.3) is relatively t_1-controllable with respect to y if and only if

$$\operatorname{rank}\left[Y_k(ih),\ k = 0, 1, \ldots, n;\ i = 0, 1, \ldots, \min\{T_{t_1-0}, m\}\right] = m.$$

Definition 4.17. System (4.1)–(4.3) is called $\mathbb{R}^n$-*controllable with respect to* x if it is $\mathbb{R}^n$-controllable with respect to x in $[t_0, t_*]$ for some $t_* > t_0$.

Definition 4.18. System (4.14), (4.15) is $\mathbb{R}^n$-*observable with respect to* x if

$$z^*(t, x_0^*, \psi^*) = z^*(t, \tilde{x}_0^*, \psi^*) \text{ a.e for } t > t_0 \Rightarrow x_0^* = \tilde{x}_0^*.$$

Theorem 4.19. The following statements are equivalent:

i) System (4.1)–(4.3) is $\mathbb{R}^n$-controllable with respect to x;

ii) System (4.14), (4.15) is $\mathbb{R}^n$-observable with respect to x;

iii) $\operatorname{rank}[X_k(ih),\ k = 1, 2, \dots, n;\ i = 0, 1, \dots, m] = n$.

5. Open Problems

The concept of hybrid system theory has not been uniquely defined yet and that is why the first open problem under consideration is the following one.

Problem 5.1. The construction of general hybrid system control and observation model.

Let us turn to systems (3.3)–(3.5). For such a system the following problems are still open.

Problem 5.2. The statement of the initial value problem.

Problem 5.3. Variation-of-constants formulae for nonstationary systems.

Problem 5.4. Exponential bound for solutions of stationary systems.

Problem 5.5. Representations of solutions of stationary systems into series of their determining equation solutions.

Definition 5.6. System (3.3)–(3.5) is called *completely* H-t_1-*controllable at* $t_1 > t_0 + h$ if for any initial data of the system there exists a control such that for the corresponding solution the following condition is valid:

$$H\begin{bmatrix} x(t_1 + t) \\ y(t_1 + t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t \geq 0.$$

Problem 5.7. Find parametric criteria for complete controllability of the system.

Using game approach [14], we can consider more general concepts of controllability like (s, p)- and $\mathbb{R}^n - (s, p)$ controllability.

For given $p \geq 0, s > p$, and $s > 0$, system (3.2) is said to be $\mathbb{R}^n$-(s, p)-*controllable with respect to* x *at time* $t_* = t_0 + s$ if for any initial data φ_0, φ and ψ_0, ψ and for any piecewise continuous r-vector function v there exists a piecewise continuous control function $u = u(\cdot)$ such that for the corresponding solutions the following condition holds:

$$x(t_*; t_* - s, \varphi_0, \varphi, u) = x(t_*; t_* - p, \psi_0, \psi, v).$$

Problem 5.8. Find parametric criteria for (s, p)- and $\mathbb{R}^n$-(s, p)-controllability and its dual observability problems.

The classical problem of dynamic control theory is a problem of stability and stabilization.

Consider DAD system (3.6) with $u(t) = 0$ for $t \geq 0$ and the corresponding characteristic equation

$$\det \begin{bmatrix} \lambda - A_{11} & -A_{12} \\ -A_{21} & 1 - A_{22}e^{-\lambda h} \end{bmatrix} = \Delta(\lambda) = 0, \quad \lambda \in \mathbb{C}.$$

Then the condition $\operatorname{Re} \lambda < 0$, for all complex numbers λ such that $\Delta(\lambda) = 0$, is necessary for each type of the following kinds of stability: asymptotic, exponential, and L_2 stability.

Problem 5.9. Investigate several kinds of stability of open system (3.3)–(3.5) and give necessary and sufficient stability conditions.

Problem 5.10. Modal control, stabilization, and reconstruction of DAD systems.

Problem 5.11. Realization of DAD systems in scale [15, 16] of dynamic systems.

6. Concluding Remarks

In the paper we have discussed the character and several examples of hybrid systems. Numerous available results in the field are given and some open problems are formulated. The methods used can be applied to 2-D hybrid discrete continuous dynamic systems. This is a subject of another paper.

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References

[1] R. Marz. Solvability of Linear Differential Algebraic Equations with Properly Stated Leading Terms, Results in Mathematics, Birkhäuser Verlag, Basel, 45:88–95, 2004.

[2] F.M. Kirillova and S.V. Strel'tsov. Necessary Optimality Conditions for Hybrid Systems, *Control Systems*, SO AN USSR, 14:24–33, 1975.

[3] A.A. Akhundov. Controllability of Hybrid Systems, *Control Systems*, SO AN USSR, 14:4–10, 1975.

[4] T.S. Trofimchuk. Controllability of Systems not Solved with Respect to Derivative, *Control Systems*, SO AN USSR, 20:75–82, 1980.

[5] A.A. Shcheglova. Observability of Degenerate Linear Hybrid Systems, *Avtomatika i Telemekhanika*, 11:86–101, 2004.

[6] J.J. Gertler, J.B. Cruz, and M. Peshkin, (Eds.), Hybrid Systems, *Prepr. 13th World Congr. IFAC*, vol. J, pp. 2753–11, 473–476, 1996.

[7] M. De la Sen. The Reachability and Observability of Hybrid Multirate Sampling Linear Systems, *Computers Math. Appl.*, 31(1):109–122, 1996.

[8] R. Vidal, A. Chiuso, S. Soato, and S. Sastry. Observability of Linear Hybrid Systems, *Hybrid Systems: Computation and Control*, LNCS, 2623:526–539, 2003.

[9] V.M. Marchenko, O.N. Poddubnaya, and Z. Zaczkiewicz. On the Observability of Linear Differential-Algebraic Systems with Delay, *IEEE Trans. Automat. Contr.*, 51(8):1387–1392, 2006.

[10] I.F. Kuz'mitskii, V.M. Marchenko, and O.N. Poddubnaya. Differential-Algebraic Systems in Rectification Production Process, *Abstr. Of Intern. Conf. Problems of Control and Applications* (May 16–20, 2005), BNTU, Minsk, pp. 152–153, 2006.

[11] S. Domek and R. Kaszyński, (Eds.), Hybrid Systems: Computation and Control, IEEE Conf. MMAR'2004, vol. 1: Control Theory, Control Engineering, Modelling and Simulation, Blazejewko, Poland, 2004.

[12] A. Van der Schaft and H. Schumacher. An Introduction to Hybrid Dynamical Systems, Springer, Berlin, 2000.

[13] V.M. Marchenko and O.N. Poddubnaya. Solution Representations and Relative Controllability of Linear Differential Algebraic Systems with Several Delays, *Doklady Mathematical Sciences*, 72(2):824–828, 2005.

[14] V.M. Marchenko. Systems of Control with After-Effect and Open Problems, *Modelling and Simulation: Proceedings of International Conference MS'2004* (27-29 April, 2004), Minsk, Belarus, pp. 21–26, 2004.

[15] V.M. Marchenko and J.-J. Loiseau. Realization of Dynamical Systems with After-effect: I. Realizability, *Differential Equations*, 42(11):1587–1595, 2006.

[16] V.M. Marchenko and J.-J. Loiseau. Realization of Dynamical Systems with After-effect: II. Controllability. Duality. Minimality, *Differential Equations*, 43(1):1–13, 2007.