

Using Chebyshev Multiwavelet Basis for Solving Integral Equation

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Abstract

In this work, we are reviewing some papers with Mallat, Meyer, Daubechies, Alpert and others. In these papers the authors constructed a wavelet basis for $L^2(\mathbb{R})$. Then with using orthonormality of Chebyshev polynomials we construct an orthonormal wavelet basis for $L^2[0, 1]$. This basis help us in solving integral equations. One characteristic of such basis functions is that after discretizing integral equation with projection methods we got systems with small dimension but with high precision.

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1. Introduction

With reviewing the work in [1], we construct a class of bases for $L^2[0, 1]$. For this we use Chebyshev polynomials. By considering $[0, 1] \subset \mathbb{R}$ and Multi-Resolution Analysis (MRA) that was used in [2–4], we present multiwavelets that have been constructed from Chebyshev polynomials. For solving the integral equation we used an orthonormal basis constructed from the mentioned multiwavelets in $L^2[0, 1]$.

2. Introducing Chebyshev Multiwavelets

By considering Chebyshev polynomials in $[-1, 1]$, the Chebyshev orthonormal functions are

$$\left\{ \sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}x, \sqrt{\frac{2}{\pi}}(2x^2 - 1), \sqrt{\frac{2}{\pi}}(4x^3 - 3x), \sqrt{\frac{2}{\pi}}(8x^4 - 8x^2 + 1), \dots \right\}, \quad (2.1)$$

similarly, the Chebyshev orthonormal functions in $[0, 1]$ are

$$\left\{ \sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}(2x - 1), \sqrt{\frac{2}{\pi}}(8x^2 - 8x + 1), \sqrt{\frac{2}{\pi}}(32x^3 - 48x^2 + 18x - 1), \right. \\ \left. \sqrt{\frac{2}{\pi}}(128x^4 - 256x^3 + 160x^2 - 32x + 1), \dots \right\}.$$

We divide $[0, 1]$ into 2^m subinterval and in any subinterval the polynomial $f(x)$ has degree less than k and is zero outside $[0, 1]$. So for a constant k V_0^k , V_1^k and V_m^k are introduced as:

$$V_0^k = \left\{ f \mid f(x) = \begin{cases} \text{polynomials with degree less than } k, & 0 < x < 1; \\ 0, & \text{otherwise,} \end{cases} \right\} \quad (2.2)$$

$$V_1^k = \left\{ f \mid f(x) = \begin{cases} \text{polynomials with degree less than } k, & 0 < x < \frac{1}{2}; \\ \text{polynomials with degree less than } k, & \frac{1}{2} < x < 1; \\ 0, & \text{otherwise,} \end{cases} \right\} \quad (2.3)$$

$$V_m^k = \left\{ f \mid f(x) = \begin{cases} \text{polynomials with degree less than } k, & \frac{j}{2^m} < x < \frac{j+1}{2^m}; \\ 0, & \text{otherwise,} \end{cases} \right\} \quad (2.4)$$

for $j = 0, 1, \dots, 2^m - 1$. From the above definitions the dimensions of V_0^k , V_1^k and V_m^k are k , $2k$ and $2^m k$ respectively. The defined spaces are nested. So

$$V_0^k \subset V_1^k \subset \dots \subset V_m^k \subset \dots \quad (2.5)$$

Suppose that W_m^k be the orthogonal complement for V_m^k , then

$$V_{m+1}^k = V_m^k \oplus W_m^k, \quad W_m^k \perp V_m^k, \quad (2.6)$$

also we have:

$$V_{m+1}^k = V_0^k \oplus W_0^k \oplus \dots \oplus W_m^k. \quad (2.7)$$

Since $\dim(V_{m+1}^k) = 2^{m+1}k$ and $\dim(V_m^k) = 2^m k$, the dimension of W_m^k is $2^m k$.

First consider the space W_0^k whose orthonormal basis is $\{\omega_1, \omega_2, \dots, \omega_k\}$. Since the basis elements of V_0^k are of degree less than k and $W_0^k \perp V_0^k$, so we have:

$$\int_0^1 \omega_j(x)x^i dx = 0, \quad i = 0, 1, \dots, k-1, \quad j = 1, 2, \dots, k, \quad (2.8)$$

in other words, $\omega_j(x)$ is piecewise polynomial that is zero outside $[0, 1]$ and is orthogonal to polynomials with lower degree. So, with considering that the dimension of W_0^k is k , we have:

$$W_0^k = \text{span} \{ \omega_1(x), \omega_2(x), \dots, \omega_k(x) \}. \quad (2.9)$$

Similarly, W_m^k with scaling and translating of basis elements of W_0^k is defined as follows:

$$W_m^k = \text{span} \left\{ \omega_{j,m}^n(x) = 2^{m/2} \omega_j(2^m x - n) \right\} \quad (2.10)$$

with $j = 1, 2, \dots, k$ and $n = 0, 1, \dots, 2^m - 1$. In this order we obtain the following nested subspaces:

$$W_0^k \subset W_1^k \subset \dots \subset W_m^k \subset \dots \quad (2.11)$$

The above process leads to

$$\overline{\bigcup_{m=0}^{\infty} W_m^k} = L^2[0, 1]. \quad (2.12)$$

For obtaining $\omega_1, \omega_2, \dots, \omega_k$ we use the following algorithm:

Algorithm:

1. Consider the functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, k$ as:

$$f_i(x) = \begin{cases} \sum_{p=0}^{k-1} a_p x^p, & 0 < x < 1; \\ (-1)^{i+k-1} \sum_{p=0}^{k-1} a_p (-x)^p, & -1 < x < 0; \\ 0, & \text{otherwise.} \end{cases}$$

Note that the first case is a linear combination of polynomials with degree less than k over $(0, 1)$ and the second case is the extension of the first case (as an odd or even function) in $(-1, 0)$.

2. Apply the orthonormality condition on f_i 's, *i.e.*,

$$\int_{-1}^1 \frac{f_i(x) f_j(x)}{\sqrt{1-x^2}} dx = \delta_{ij}, \quad 1 \leq i, j \leq k.$$

3. Vanishing moments condition:

$$\int_{-1}^1 \frac{f_i(x) x^j}{\sqrt{1-x^2}} dx = 0, \quad j = 0, 1, \dots, i+k-2.$$

We calculate the above steps for $k = 1, 2, 3$. For larger k 's we can use a similar process.

Case $k = 1$:

Considering the above algorithm for $k = 1$ we have $i = 1$, the function $f_1(x)$ is defined as

$$f_1(x) = \begin{cases} a_0, & 0 < x < 1; \\ -a_0, & -1 < x < 1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.13)$$

By applying conditions 2 and 3 of the algorithm, we have $a_0 = \sqrt{\frac{1}{\pi}}$. So the basis of W_0^1 is calculated as

$$\omega_1(x) = f_1(2x - 1) = \begin{cases} -\sqrt{\frac{1}{\pi}}, & 0 < x < \frac{1}{2}; \\ \sqrt{\frac{1}{\pi}}, & \frac{1}{2} < x < 1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.14)$$

Consequently, $V_1^1 = V_0^1 \oplus W_0^1$ as a subspace of $L^2[0, 1]$ is produced as follows:

$$V_1^1 = \text{linear span} \left\{ \sqrt{\frac{1}{\pi}}, \begin{cases} -\sqrt{\frac{1}{\pi}}, & 0 < x < \frac{1}{2}; \\ \sqrt{\frac{1}{\pi}}, & \frac{1}{2} < x < 1. \end{cases} \right\} \quad (2.15)$$

The above basis is an orthonormal basis for the space V_1^1 .

Case $k = 2$

Considering the above algorithm for $k = 2$, we have $i = 1, 2$, the function $f_1(x)$ and $f_2(x)$ are defined as

$$f_1(x) = \begin{cases} a_0 + a_1x, & 0 < x < 1; \\ a_0 - a_1x, & -1 < x < 1; \\ 0, & \text{otherwise;} \end{cases} \quad (2.16)$$

$$f_2(x) = \begin{cases} a_0 + a_1x, & 0 < x < 1; \\ -a_0 + a_1x, & -1 < x < 1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.17)$$

By applying conditions 2 and 3 of the algorithm, the space $V_1^2 = V_0^2 \oplus W_0^2$ as a subspace of $L^2[0, 1]$ is produced as

$$V_1^2 = \text{linear span} \left\{ \sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}(2x - 1), \omega_1(x), \omega_2(x) \right\} \quad (2.18)$$

with

$$\omega_1(x) = \frac{\sqrt{2}}{\sqrt{\pi^3 - 8\pi}} \begin{cases} \pi(2x - 1) + 2, & 0 < x < \frac{1}{2}; \\ -\pi(2x - 1) + 2, & \frac{1}{2} < x < 1; \end{cases}$$

$$\omega_2(x) = \frac{1}{\sqrt{\pi^3 - 8\pi}} \begin{cases} 4(2x - 1) + \pi, & 0 < x < \frac{1}{2}; \\ 4(2x - 1) - \pi, & \frac{1}{2} < x < 1. \end{cases}$$

The above basis is an orthonormal basis for the space V_1^2 .

Case $k = 3$

Similarly for $k = 3$, the space $V_1^3 = V_0^3 \oplus W_0^3$ as a subspace of $L^2[0, 1]$ is produced as

$$V_1^3 = \text{linear span} \left\{ \sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}(2x - 1), \sqrt{\frac{2}{\pi}}(8x^2 - 8x + 1), \omega_1(x), \omega_2(x), \omega_3(x) \right\} \quad (2.19)$$

with

$$\begin{aligned} \omega_1(x) &= \frac{1}{\sqrt{\pi(1441792 - 295560\pi^2 + 15147\pi^3)}} \\ &\times \begin{cases} -(2048 - 207\pi^2) - 36\pi(2x - 1) - 12(27\pi^2 - 256)(2x - 1), & 0 < x < \frac{1}{2}; \\ (2048 - 207\pi^2) - 36\pi(2x - 1) + 12(27\pi^2 - 256)(2x - 1), & \frac{1}{2} < x < 1, \end{cases} \\ \omega_2(x) &= \frac{\sqrt{2}}{\sqrt{9\pi^3 - 88\pi}} \begin{cases} 2 + 3\pi(2x - 1) + 8(2x - 1)^2, & 0 < x < \frac{1}{2}; \\ 2 - 3\pi(2x - 1) + 8(2x - 1)^2, & \frac{1}{2} < x < 1, \end{cases} \\ \omega_3(x) &= \frac{\sqrt{2}}{\sqrt{1683\pi^3 - 16384\pi}} \\ &\times \begin{cases} -12\pi - 128(2x - 1) - 30\pi(2x - 1)^2, & 0 < x < \frac{1}{2}; \\ 12\pi - 128(2x - 1) + 30\pi(2x - 1)^2, & \frac{1}{2} < x < 1. \end{cases} \end{aligned}$$

The above basis is an orthonormal basis for the space V_1^3 . For greater k ($k = 4, 5, \dots$) we can act in a similar way.

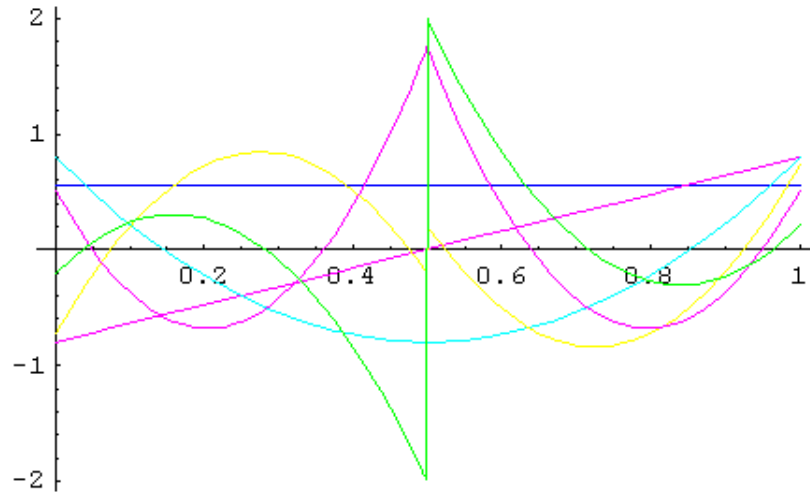
3. Applying the Method

Consider the following integral equation:

$$u(t) - \int_0^1 k(t, s)u(s) ds = f(t), \quad 0 \leq t \leq 1, \quad (3.1)$$

where $f(t)$ and $k(t, s)$ are known functions and $u(t)$ is the unknown function to be determined. Suppose that the equation (20) has a solution in $L^2[0, 1]$. With considering that $\bigcup_{m=0}^{\infty} V_m^k = L^2[0, 1]$, the solution of the integral equation ($u(t)$) is approximated in the subspaces of $L^2[0, 1]$, namely V_m^k , $m = 0, 1, \dots$, $k = 2, 3, \dots$ as follows:

$$u(t) = \sum_{i=1}^n c_i b_i(t), \quad (3.2)$$

Figure 1: Orthonormal basis of elements for V_1^3 .

where $\{b_i(t)\}_{i=1}^n$ is an orthonormal basis of V_m^k and $n = 2^m k$,

$$\sum_{i=1}^n c_i b_i(t) - \int_0^1 \left[\sum_{i=1}^n \sum_{j=1}^n k_{ij} b_i(t) b_j(s) \right] \left[\sum_{k=1}^n c_k b_k(s) \right] ds = \sum_{i=1}^n c_i b_i(t). \quad (3.3)$$

By multiplying both sides of the above equation by $\frac{b_j(t)}{\sqrt{t-t^2}}$ and integrating from 0 to 1 and using the orthonormality of the basis elements, we have the following system:

$$c_i - \sum_{k=1}^n c_k \sum_{j=1}^n k_{ij} b_{jk} = f_i, \quad 1 \leq i \leq n. \quad (3.4)$$

The matrix form of the above system is:

$$(I - KB)C = F \quad (3.5)$$

where $I_{n \times n}$ is an identity matrix and the elements of $K_{n \times n}$, $B_{n \times n}$ and $F_{n \times 1}$ are as follows:

$$\begin{aligned} k_{ij} &= \int_0^1 \int_0^1 \frac{k(t, s) b_i(s) b_j(s)}{\sqrt{t-t^2} \sqrt{s-s^2}} ds dt, \\ b_{ij} &= \int_0^1 b_i(t) b_j(t) dt, \\ f_j &= \int_0^1 \frac{f(t) b_j(t)}{\sqrt{t-t^2}} dt. \end{aligned}$$

V_m^k	Example 1	Example 2
V_0^2	8.4×10^{-2}	8.9×10^{-2}
V_0^3	6×10^{-3}	6.1×10^{-3}
V_0^4	3.8×10^{-4}	3.8×10^{-4}
V_0^5	1.9×10^{-5}	1.9×10^{-5}
V_1^2	1.7×10^{-2}	1.5×10^{-2}
V_1^3	2.3×10^{-4}	7.7×10^{-4}
V_1^4	1.5×10^{-5}	2.4×10^{-5}

Table 1: Numerical results for Examples 1 and 2.

4. Numerical Examples

Example 4.1. In this example, we solve

$$u(t) - \int_0^1 t^2 e^{s(t-1)} u(s) ds = e^t \quad (4.1)$$

with the exact solution $u(t) = e^t$. The numerical results are presented in Table 1.

Example 4.2. In this example, we solve

$$u(t) - \int_0^1 s^2 \sqrt{t-t^2} u(s) ds = e^t - (e-2)\sqrt{t-t^2} \quad (4.2)$$

with the exact solution $u(t) = e^t$. The numerical results are presented in Table 1.

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