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Using Quadratic B-Spline Scaling Functions for Solving Integral Equations

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Abstract

In this article, quadratic semiorthogonal B-spline scaling functions together with their dual functions are developed to approximate the solutions of linear Fredholm integral equations of the second kind. First, the quadratic B-spline scaling functions and their properties are presented; these properties are then utilized to reduce the computation of integral equations to some algebraic equations. The method is computationally attractive, some numerical examples are presented to support our work.

AMS Subject Classification:
Keywords: Semiorthogonal, scaling function, quadratic B-spline, integral equation.

1. Introduction

In the present paper, we apply compactly supported quadratic semiorthogonal (SO) B-spline scaling functions to solve the second kind linear Fredholm integral equations of the form:

$$y(x) - \int_0^1 k(x,t)y(t)\,dt = g(x), \quad 0 \le x \le 1, \tag{1.1}$$

where $k(x,t)$ and $g(x)$ are known functions and $y(x)$ is an unknown function to be determined.

Our method consist of reducing (1.1) to a set of algebraic equations by expanding the unknown function as quadratic B-spline scaling functions, with unknown coefficients. The properties of the quadratic B-spline scaling functions are then utilized to evaluate the unknown coefficients.

2. B-Spline Scaling Functions on [0, 1]

Scaling functions can be used to expand any function in $L^2(\mathbb{R})$. These functions are defined on the entire real line, so that they could be outside the domain of the problem. In order to avoid this, compactly supported spline scaling functions, constructed for the bounded interval [0, 1], have been taken into account in this article.

When semiorthogonal B-spline scaling functions of order m are used, the condition

$$2^{j_0} \geq m, \tag{2.1}$$

must be satisfied in order to have at least one complete inner scaling function. In this paper, we will use quadratic B-spline, $m = 3$ cardinal B-spline function. From (2.1), the third-order B-spline lowest level, which must be an integer, is determined as $j_0 = 2$.

The third-order B-spline scaling functions are given by

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j - k)^2, & k \leq x_j \leq k+1; \\ \frac{3}{4} - \left((x_j - k) - \frac{3}{2}\right)^2, & k+1 \leq x_j \leq k+2; \\ \frac{1}{2}((x_j - k) - 3)^2, & k+2 \leq x_j < k+3, \quad k = 0, \ldots, 2^j - 3; \\ 0, & \text{otherwise}; \end{cases} \tag{2.2}$$

with the respective left- and right-hand side boundary scaling functions

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j - k)^2, & 0 \leq x_j \leq 1, \quad k = -2; \\ 0, & \text{otherwise}; \end{cases} \tag{2.3}$$

$$\phi_{j,k}(x) = \begin{cases} \frac{3}{4} - \left((x_j - k) - \frac{3}{2}\right)^2, & k+1 \leq x_j \leq k+2; \\ \frac{1}{2}((x_j - k) - 3)^2, & k+2 \leq x_j \leq k+3, \quad k = -1; \\ 0, & \text{otherwise}; \end{cases} \tag{2.4}$$

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j - k)^2, & k \leq x_j \leq k+1; \\ \frac{3}{4} - \left((x_j - k) - \frac{3}{2}\right)^2, & k+1 \leq x_j \leq k+2; \quad k = 2^j - 2; \\ 0, & \text{otherwise}; \end{cases} \tag{2.5}$$

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}((x_j - k) - 3)^2, & k+2 \le x_j \le k+3, \quad k = 2^j - 1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

The actual coordinate position x is related to x_j by $x_j = 2^j x$.

3. Function Approximation

For any fixed positive integer M, a function $f(x)$ defined over [0, 1] may be presented by B-spline scaling functions as

$$f(x) = \sum_{k=-2}^{2^M-1} s_k \phi_{M,k} = S^T \Phi_M \tag{3.1}$$

where

$$\begin{aligned} S &= \left[s_{-2}, s_{-1}, \ldots, s_{2^M-1}\right], \\ \Phi_M &= \left[\phi_{M,-2}, \phi_{M,-1}, \ldots, \phi_{M,2^M-1}\right], \end{aligned} \tag{3.2}$$

with

$$s_k = \int_0^1 f(x)\tilde{\phi}_{M,k}(x)\,dx, \quad k = -2, -1, \ldots, 2^M - 1, \tag{3.3}$$

where $\tilde{\phi}_{M,k}(x)$ are dual functions of $\phi_{M,k}(x)$. These can be obtained by linear combinations of $\phi_{M,k}(x),\ k = -2, -1, \ldots, 2^M - 1$ as follows. Let $\tilde{\Phi}_M$ be the dual functions of Φ_M given by

$$\tilde{\Phi}_M = \left[\tilde{\phi}_{M,-2}, \tilde{\phi}_{M,-1}, \ldots, \tilde{\phi}_{M,2^M-1}\right]. \tag{3.4}$$

Using (3.2) and (3.4), we get

$$\int_0^1 \tilde{\Phi}_M \Phi_M^T\,dx = I_1, \tag{3.5}$$

where I_1 is $(2^M + 2) \times (2^M + 2)$ identity matrix. Let

$$P_M = \int_0^1 \Phi_M \Phi_M^T\,dx. \tag{3.6}$$

The entry $(P_M)_{i,j}$ of the matrix P_M in (3.6) is calculated from

$$\int_0^1 \phi_{M,i}(x)\phi_{M,j}(x)\,dx. \tag{3.7}$$

For example for $M = 3$, using (2.2)–(2.6) and (3.1) we get:

$$\int_0^1 \Phi\Phi^T dx$$

$$= P_3 = \begin{pmatrix} \frac{1}{160} & \frac{13}{960} & \frac{1}{960} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{13}{960} & \frac{60}{960} & \frac{13}{480} & \frac{1}{960} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{960} & \frac{13}{960} & \frac{11}{160} & \frac{13}{480} & \frac{1}{960} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{960} & \frac{13}{960} & \frac{11}{160} & \frac{13}{480} & \frac{1}{960} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{960} & \frac{13}{960} & \frac{11}{160} & \frac{13}{480} & \frac{1}{960} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{960} & \frac{13}{960} & \frac{11}{160} & \frac{13}{480} & \frac{1}{960} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{960} & \frac{13}{960} & \frac{11}{160} & \frac{13}{480} & \frac{1}{960} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{960} & \frac{13}{960} & \frac{11}{160} & \frac{13}{480} & \frac{1}{960} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{960} & \frac{13}{480} & \frac{60}{960} & \frac{13}{960} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{960} & \frac{13}{960} & \frac{1}{160} \end{pmatrix}. \tag{3.8}$$

From (3.5) and (3.6), we get

$$\tilde{\Phi}_M = (P_M)^{-1}\Phi_M. \tag{3.9}$$

4. Second Kind Fredholm Integral Equations

In this section, we solve linear Fredholm integral equation of the second kind of the form (1.1) by using B-spline scaling functions. For this we use (2.6) to approximate $y(x)$,

$$y(x) = C_2^T \Phi(x), \tag{4.1}$$

where $\Phi(x)$ is defined in (3.3), and C_2 is $(2^{M+1}+2)\times 1$ unknown vector defined similarly to S in (2.6). We also expand $g(x)$ and $k(x,t)$ by B-spline dual scaling functions $\tilde{\Phi}$ defined as in (3.3) as

$$g(x) = C_1^T \tilde{\Phi}, \quad k(x,t) = \tilde{\Phi}^T(t)\Theta\tilde{\Phi}(x), \tag{4.2}$$

where

$$\Theta_{(i,j)} = \int_0^1 \left[\int_0^1 k(x,t)\Phi_i(t)\,dt \right] \Phi_j(x)\,dx. \tag{4.3}$$

From (4.1) and (4.2) we get

$$\begin{aligned} \int_0^1 k(x,t)y(t)\,dt &= \int_0^1 \tilde{\Phi}^T(t)\Theta\tilde{\Phi}(x)C_2^T\Phi(t)\,dt \\ &= C_2^T\Theta\tilde{\Phi}(x) \end{aligned} \tag{4.4}$$

By applying (4.1)–(4.4) in equation (1.1) we have:

$$C_2^T\Phi(x) - C_2^T\Theta\tilde{\Phi}(x) = C_1^T\tilde{\Phi}(x) \tag{4.5}$$

By multiplying equation (4.5) by $\Phi^T(x)$ and integrating with respect to x we get

$$C_2^T P_M - C_2^T\Theta = C_1^T. \tag{4.6}$$

and so $C_2^T = C_1^T(P - \Theta)^{-1}$.

Now we can calculate the solution with $y(x) = C_2^T\Phi(x)$.

5. Numerical Results

In this section we use some numerical examples to test the method and show the efficiency and accuracy of the method. The absolute errors at some different points are presented in Table 1.

x	Ex. 1, $M=2$	Ex. 1, $M=3$	Ex. 2, $M=2$	Ex. 2, $M=3$
0.0	1.49594×10^{-4}	1.80032×10^{-5}	8.92857×10^{-4}	1.08722×10^{-4}
0.1	4.46651×10^{-5}	1.30284×10^{-5}	2.85714×10^{-4}	7.2382×10^{-5}
0.2	1.09901×10^{-4}	1.19936×10^{-5}	5.71429×10^{-4}	5.52499×10^{-5}
0.3	1.67663×10^{-4}	1.00466×10^{-5}	7.5×10^{-4}	4.93723×10^{-5}
0.4	1.30998×10^{-4}	2.33899×10^{-5}	4.64286×10^{-4}	9.33736×10^{-5}
0.5	3.26269×10^{-5}	1.88115×10^{-6}	9.71445×10^{-17}	5.55112×10^{-17}
0.6	1.18775×10^{-4}	2.79725×10^{-5}	4.64286×10^{-4}	9.33736×10^{-5}
0.7	2.47169×10^{-4}	1.81121×10^{-5}	7.5×10^{-4}	4.93723×10^{-5}
0.8	2.22559×10^{-4}	1.9016×10^{-5}	5.71429×10^{-4}	5.52499×10^{-5}
0.9	1.33422×10^{-4}	3.03243×10^{-5}	2.85714×10^{-4}	7.2382×10^{-5}
1.0	3.97292×10^{-4}	4.92832×10^{-5}	8.92857×10^{-4}	1.08722×10^{-4}

Table 1: Numerical results for Examples 5.1 and 5.2 with $M = 2, 3$.

Example 5.1. For the first example, we solve (1.1) with $k(x,t) = xte^{xt}$ and $g(x) = e^x - x - \frac{x(1+xe^{x+1})}{(1+x)^2} - \frac{2-e^x(x^2-2x+2)}{x^2}$ with the exact solution $y(x) = e^x - x$. The absolute errors at different points with $M = 2, 3$ are presented in columns 2, 3 of Table 1.

Example 5.2. As the second example, we solve (1.1) with $k(x,t) = 2(x+t)^2$ and $g(x) = x^3 - \frac{1}{2}x^2 - \frac{4}{5}x - \frac{1}{3}$ with the exact solution $y(x) = x^3$. The absolute errors at different points with $M = 2, 3$ are presented in columns 4, 5 of Table 1.

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