International Journal of Lakes and Rivers.
ISSN 0973-4570 Volume 17, Number 2 (2024), pp. 125-133

http://www.ripublication.com/ijlr.htm

Modeling and Simulation of a Single Reservoir System and Its Operations

Kaushal B. Patel*

Department of Mathematics
Veer Narmad South Gujarat University, Surat, Gujarat, India

Abstract

Reservoirs are fundamental elements of water-management systems, serving key functions such as water conservation, flood control, water treatment, and supporting aquatic environments. Effective reservoir management requires a dynamic process to identify uncertainties affecting behavior and minimize their impact on performance.

We aim to develop a mathematical model of a reservoir compartment system and its operations. Reservoir operations are represented by a set of simultaneous non-linear equations. The equilibrium of this system of equations is derived, and the reproduction number is calculated. We discuss local and global stability analyses with variations in parameters. The derived model is then simulated to describe the typical operation of a reservoir system, providing a stable utility solution for the reservoir.

Keywords: Basic Reproduction number, Compartment model, Non-linear equations, Reservoir system, Stability analysis, Simulation

1. Introduction

A water reservoir is an enclosed area for storing water, usually constructed by building a dam across water resources using natural or constructed depressions. Water is collected upstream of a river. Reservoirs play a crucial role in storing water during the rainy season, controlling the flow of floods, and gradually releasing water during periods of lower flow. They are used for drinking, irrigation, industrial purposes, fishing, boating, and other recreational activities. Additionally, reservoirs are used for generating electricity.

In the western region of India, rivers are generally full during the monsoon, but carry very little water during the rest of the year. Hence, proper reservoir operation management is essential to meet water demand throughout the year. In summer, water demand increases, and evaporation due to higher temperatures causes water loss. Therefore, managing reservoir water levels to meet demand is critical.

Yeh discussed several approaches for optimal reservoir operation along with their limitations (Yeh, 1985). Vedula and Rogers introduced a non-fuzzy multi-objective optimization approach (Vedula and Rogers, 1981). Nagesh D. and Reddy M. used evolutionary algorithms and swarm intelligence methods for water resource management (D. Nagesh and M. Reddy, 2020). A model based on mixed-integer programming techniques, catering to different needs through step-by-step operation of multi-reservoirs, was developed by Mohammad Heydari, Faridah Othman, and Kourosh Qaderi (2015). Florian T. Bessler, Dragan A. Savic, and Godfrey A. Walters (2013) proposed a general operating policy for water supply systems using data mining. Dingzhi Peng, Shenglian Guo, Pan Liu, and Ting Liu (2016) provided a reservoir storage curve using remote sensing data for multipurpose reservoir operation and strategic risk management.

Most of the work related to reservoir operation employs linear programming techniques. In this study, we consider reservoir components such as the environment, irrigation, daily use, industrial use, and sewage, forming a system of non-linear differential equations. We also discuss various stability analyses to observe the behavior of reservoir compartments given the parameters.

2. Mathematical Model

The proposed mathematical model consists of four compartments: Reservoir Level, Daily Demand (Irrigation, Daily Usage, and Industrial Usage), Evaporation, and Sewage. The upstream river flow charges the reservoir compartment. To meet water demand, water is released from the reservoir for various purposes such as irrigation, daily needs, and industrial uses. The sewage compartment accounts for water after utilization, including polluted water. This waste or utilized water is either diverted to another river or returned to the same river downstream. $R, I, E,$ and S represent the cubic volume of water in the Reservoir, Irrigation, Evaporation, and Sewage compartments, respectively.

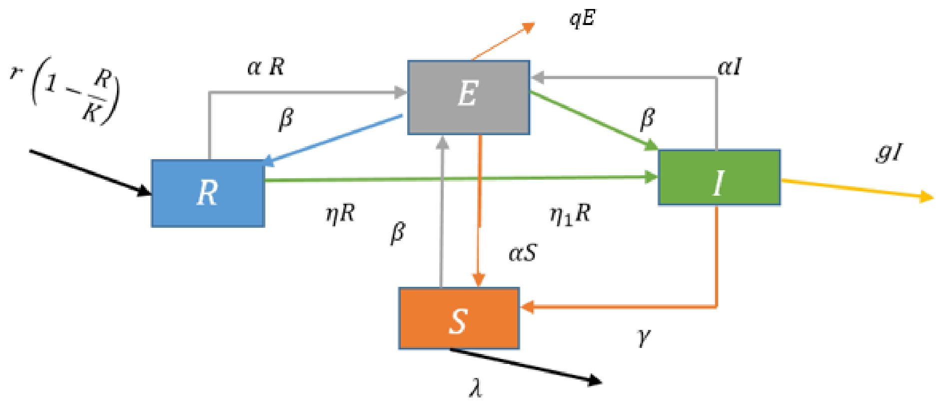


Figure 1 schematic representation of variable and components dependency in the mathematical model

2.1 Assumption:

- The input from the river and upstream remains constant during the period $0 \leq t \leq T$.

- Daily water usage for various purposes remains constant during the period $0 \le t \le T$.
- The evaporation rate remains constant during the period $0 \le t \le T$.
- Water loss due to humidity remains constant during the period $0 \le t \le T$.
- Seepage losses remain constant during the period $0 \le t \le T$.

The system is governed by the following equations:

$$\frac{dR}{dt} = rR\left(1 - \frac{R}{K}\right) + \beta E - \alpha RE - \eta RI \qquad (\mathbf{1})$$

$$\frac{dE}{dt} = \alpha(RE + IE + SE) - q\,E \qquad (\mathbf{2})$$

$$\frac{dI}{dt} = \eta_1 RI + \beta E - \alpha\,IE - g\,I \qquad (\mathbf{3})$$

$$\frac{dS}{dt} = \gamma I + \beta E - \alpha SE - \lambda S \qquad (\mathbf{4})$$

$$\frac{d(R+E+I+S)}{dt} = rR\left(1 - \frac{R}{K}\right) + (3\beta - q)E - (\eta - \eta_1)RI - (g - \gamma)I - gI - \lambda S$$
$$\ge rR\left(1 - \frac{R}{K}\right) - \lambda S \ge 0$$

Where $R(0) = E(0) = I(0) = S(0) \ge 0$

Table 1: The parameter values

Parameter		**Value**
r	Input of water into the reservoir from the river	2
α	Evaporation rate from the reservoir, irrigation, and sewage compartments	1
β	Rate of incoming water into the reservoir, irrigation, and sewage due to rain	2.32
η	Rate of water supplied to the daily demand compartment for irrigation, daily uses, and industrial uses	2.6
η_1	Rate of water received from the reservoir for irrigation, daily uses, and industrial uses after losses due to seepage and evaporation	2.3
g	Utilization rate of water	10.2
γ	Rate of wastewater drained into the sewage	4.6
λ	Rate of sewage water drainage	4.4
K	Maximum capacity of the reservoir	6
q	Losses due to humidity	4.4

3. Boundedness of Solutions:

We analyze the solution of the model to ensure it is both non-negative and bounded:

The Set $\Omega = \{(R, E, I, S) / 0 \le R \le K; 0 \le E \le E^*; 0 \le I \le I^*, 0 < S \le S^*\}$

Where $E^* = \frac{4rK}{\alpha K - 4\beta}$. $I^* = \frac{4r\beta K}{(\alpha K - 4\beta)(g - \eta_1 K)}$, $S^* = \frac{4r\beta K(\gamma + g - \eta K)}{\lambda(\alpha K - 4\beta)(g - \eta_1 K)}$

From equation (1), $\frac{dR}{dt} = 0$

$$\frac{dR}{dt} \leq rR\left(1 - \frac{R}{K}\right)$$

$$0 \leq R \leq K$$

From equation (1), $\frac{dR}{dt} = 0$

$$r\,R\left(1 - \frac{R}{K}\right) - \alpha RE + \beta E \geq 0$$

$$rK - \frac{\alpha KE}{4} + \beta E \geq r\,R\left(1 - \frac{R}{K}\right) - \alpha RE + \beta E \geq 0$$

$$rK - \frac{E(\alpha K - 4\beta)}{4} \geq 0$$

$$E \leq \frac{4rK}{\alpha K - 4\beta}$$

From equation (3), $\frac{dI}{dt} = 0$

$$\eta_1 RI + \beta E - gI \geq \frac{dI}{dt} = 0$$

$$\eta_1 KI + \beta E - gI \geq \eta_1 RI + \beta E - gI \geq 0$$

$$\therefore I \leq \frac{\beta E}{g - \eta_1 K}$$

$$\therefore I \leq \frac{4r\beta K}{(\alpha K - 4\beta)(g - \eta_1 K)}$$

From equation (4), $\frac{dS}{dt} = 0$

$$\gamma \mathrm{I} - \alpha\,\mathrm{SE} + \beta\mathrm{E} - \lambda\mathrm{S} = 0$$

$$\therefore \gamma\mathrm{I} + \beta\mathrm{E} - \lambda\mathrm{S} \geq 0$$

$$\frac{\gamma\beta E}{g - \eta K} + \beta E - \lambda S \geq \gamma\mathrm{I} + \beta\mathrm{E} - \lambda\mathrm{S} \geq 0$$

$$S \leq \frac{\beta(\gamma + g - \eta K)\,E}{\lambda(g - \eta K)}$$

$$\therefore S \leq \frac{4r\beta K(\gamma + g - \eta K)}{\lambda(\alpha K - 4\beta)(g - \eta_1 K)}$$

3.1 Equilibrium point:

For the simplest case, the equilibrium points are (K, 0, 0, 0) and (3.7963, 0.1816, 0.2554, 0.3483), with the latter being calculated numerically.

For, the local stability

$$J=\begin{bmatrix} r\left(1-\frac{2R}{k}\right)-\alpha E-\eta I & -\alpha R+\beta & -\eta R & 0 \\ \alpha E & \alpha(R+I+E)-q & \alpha E & \alpha E \\ \eta_1 I & -\alpha I+\beta & \eta_1 R-\alpha E-g & 0 \\ 0 & -\alpha S+\beta & \gamma & -\alpha E-\lambda \end{bmatrix}$$

The free equilibrium point does not provide information about the stability of the solution. Therefore, we find the eigenvalues of the Jacobian at equilibrium point E_2, which are $-4.7512, -0.5053,$ and $-1.1759 \pm 2.3075i$. All eigenvalues are negative or have negative real parts, indicating that the system is locally stable.

To assess asymptotic stability, we calculate the basic reproduction number using the next-generation matrix method (*Diekmann et al., 2009; Hefferman et al., 2005; Driessche et al., 2002*). The next-generation matrix is FV^{-1}, where F and V are the Jacobian matrices of Σ and Λ, evaluated for reservoir, evaporation, daily use, irrigation, and sewage at the equilibrium point.

Next, we analyze the stability at the equilibrium point.

Let $X'=(R,E,I,S)'$

$$X'=\Im(X)-v(X)$$

$$\Im(X)=\begin{bmatrix} rR\left(1-\frac{R}{K}\right) \\ \alpha(R+I+S) \\ \eta_1 RI \\ 0 \end{bmatrix},\ v(X)=\begin{bmatrix} \alpha RE-\beta E+\eta RI\alpha EI-\beta E+gI \\ -\gamma I+\alpha SE-\beta E+\lambda S\alpha RE-\beta E+\eta RI \\ \alpha RE-\beta E+\eta RI \\ \alpha RE-\beta E+\eta RI \end{bmatrix}$$

Where $\Im(X)$ denotes new water entering the system and $v(X)$ denotes water transferring from one compartment to another.

The derivatives of $\Im(X)$ and $v(X)$ evaluated at the equilibrium point provide the matrices F and V, each of the order 4×4, defined as follows:

$$F=\left[\frac{\partial \Im}{\partial X_j}\right] \text{ and } V=\left[\frac{\partial v}{\partial X_j}\right] \text{ j=1, 2, 3, 4.}$$

$$\text{So, } F=\begin{bmatrix} r\left(1-\frac{2R}{K}\right) & 0 & 0 & 0 \\ \alpha E & \alpha(R+I+S) & \alpha E & \alpha E \\ \eta_1 I & 0 & \eta_1 R & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \alpha E + \eta I & \alpha R - \beta & \eta R & 0 \\ 0 & q & 0 & 0 \\ 0 & \alpha I - \beta & \alpha E + g & 0 \\ 0 & \alpha E - \beta & -\gamma & \alpha E - \lambda \end{bmatrix}$$

V is non-singular matrix.

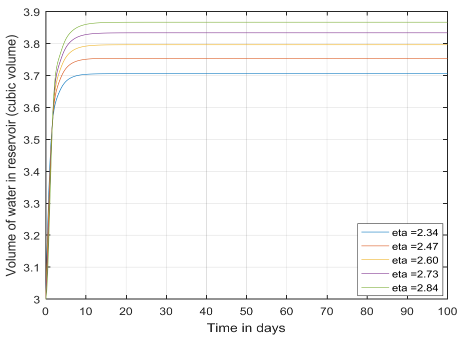
$$V^{-1} = \begin{bmatrix} \frac{1}{\eta I + \alpha E} & \frac{\alpha \eta R I - \alpha^2 R E + \alpha \beta E - \alpha g R - \beta \eta R + \beta g}{q(\alpha E + g)(\alpha E + \eta I)} & -\frac{\eta R}{(\alpha E + g)(\alpha E + \eta I)} & 0 \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & -\frac{\alpha I - \beta}{q(\alpha E + g)} & \frac{1}{(\alpha E + g)} & 0 \\ 0 & \frac{\alpha^2 E S + \alpha \gamma I - \alpha \beta E + \alpha g S - \beta g - \beta \gamma}{q(\alpha E + g)(\alpha E + \lambda)} & \frac{\gamma}{(\alpha E + g)(\alpha E + \lambda)} & \frac{1}{\alpha E + \lambda} \end{bmatrix}$$

The first equilibrium has no significant meaning, so we discard that case. Thus, in the second case, the basic reproduction number R_0 of the matrix $AV^{-1}$is given by $R_0 = 0.3424 < 1$**.**

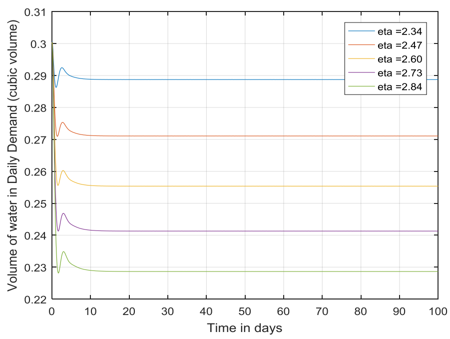
Theorem: (Stability at R_0). If R_0 <1 then equilibrium point is local asymptotically stable.

The system is globally stable if $\det(I - FV^{-1}) > 0$. The reservoir model is globally stable as $\det(I - FV^{-1}) = 1 - R_0 = 1 - 0.4973 = 0.5027$ [9].
The impact on the system on varying the parameter is also observed. Which is as follows

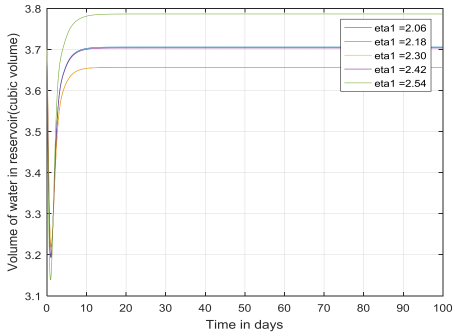
4. Simulation and Interpretation



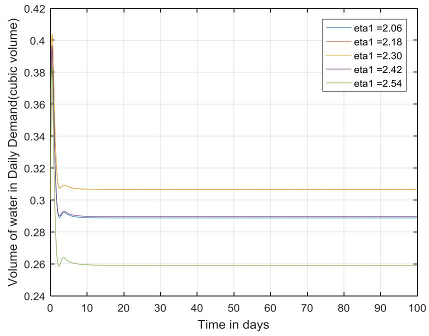
(a)



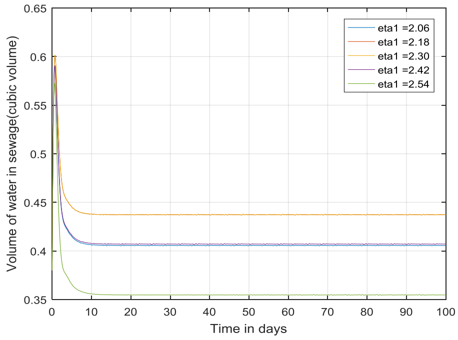
(b)



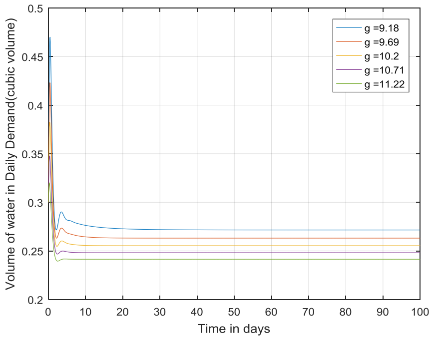
(c)



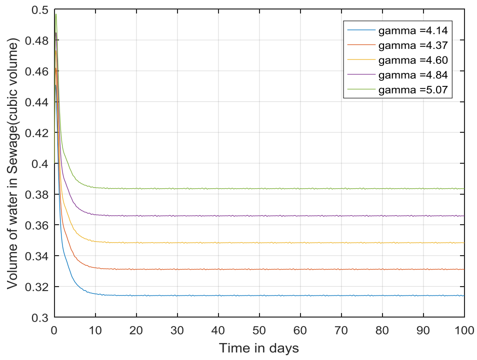
(d)



(e)



(f)



(g)

Figure 2: a) Effect of the rate of supply to daily demand (η) on the volume of water in the reservoir (b) Daily demand (c) Effect of the rate of water received by the daily demand compartment (η_1) on the volume of water in the reservoir (d) Daily demand compartment (e) Effect of the rate of supply on the volume of sewage (f) Effect of the utilization factor on daily demand (g) Effect of the rate of drain water into sewage on the volume of water in sewage

By simulating the reservoir, we observe significant effects of varying water supply for daily uses on maintaining the reservoir level. Initially, the water level rises to 3.8 cubic volumes, becoming steady after the 15th day. The volume of the reservoir increases by 3% to 5% when the supply of water for daily use is reduced by 5%. Conversely, increasing the water supply rate by 5% decreases the reservoir volume by 3%.

An increase in the rate of water supply for daily use by 10% results in a 7% drop in the reservoir level and a 2% increase in the sewage compartment, indicating that excess water for utilization may lead to wastage. We notice that every 5% variation in sewage volume corresponds to a 9% to 10% change in the volume rate of supply for daily uses. The utilization rate has a minor effect on daily water use: varying the utilization rate by 1.5% of the total volume increases the drain water into sewage by 1%.

5. Sensitivity Analysis:

Table – 2 represents the he sensitivity analysis for the parameters. With the use of $\gamma_p^Q = \frac{\partial Q}{\partial p} \cdot \frac{p}{Q}$, obtain the normalized sensitivity index of the parameters where p denotes the model parameter.

Table 2: Sensitivity Analysis

Parameter	Value	Parameter	Value
r	-	β	+
η	+	γ	+
η_1	+	λ	_
α	+	g	_

Sensitivity analysis suggests the rate of water enters into the reservoir, and the supply of water for daily demand makes a significant effect for maintaining the reservoir level.

6. Conclusion

A non-linear mathematical model for the reservoir system has been formulated. The stability of the water supply through the reservoir is analyzed using the basic reproduction number, calculated with numerical data. The basic reproduction number is 0.3434, indicating that controlling the water supply to the daily demand compartment will maintain the water level in the reservoir. Additionally, changes in various compartments are observed when the supply rate to the daily demand compartment is altered. Therefore, the supply rate should be controlled to ensure stability.

References:

[1] Mohammad H., Othman F., and Kourosh Q., 2015. Developing Optimal Reservoir Operation for Multiple and Multipurpose Reservoirs Using Mathematical Programming, Hindawi Publishing Corporation, Mathematical Problems in Engineering DOI: http://dx.doi.org/10.1155/2015/435752.

[2] Florian T. Bessler, Dragan A. Savic and Godfrey W. A., 2003. Water Reservoir Control with Data Mining, DOI: 10.1061/(ASCE) 0733-9496-2003:129:1-(26).

[3] Kumar M., Sathans, 2019. Level management in connected water reservoirs employing curve fitting based PID control scheme, the Fourth International Conference on Communication and Electronics Systems (ICCES-2019), IEEE Xplore.

[4] Jiri V., Petr D. and Martin M., Modelling and Simulation of Water Tank, ECMS2014

[5] Wai-See M., Cohon J. L. and ReVelle C. S., 1986. A Programming Model for Analysis of the Reliability, Resilience and Vulnerability of a Water Supply Reservoir, Water Resources Research, Vol. 22, No.4, pages-489-498.

[6] Reddy M., Kumar D.N., 2020. Evolutionary algorithms, swarm intelligence methods, and their applications in water resources engineering: a state-of-the-art review, H_2Open Journal 3(1): 135-188, https://doi.org/10.2166/h2oj.2020.128.

[7] Peng D., Guo S., Liu P.and Liu T., 2006. Reservoir Storage curve Estimation Based on Remote Sensing Data, Journel of Hydrologic Engineering –Vol. 11 Issue-2.

[8] Shah N.H., Satia M.H., Yeolekar B. M., 2017 Optimum Control on spread of Pollutants through Forest. Resources, Applied Mathematics, Vol.-8, Page-607-620.

[9] Shah N.H., Satia M.H., Yeolekar B. M., 2017. Optimal Control on depletion of Green Belt due to Industries, Advances in Dynamical Systems and Applications, Research India Publications, ISSN 0973-5321, Volume 12, Number 3 pp. 217-232.

[10] Diekmann, O., Heesterbeek, J. A. P., & Roberts, M. G. (2009). The construction of next-generation matrices for compartmental epidemic models, *Journal of the Royal Society Interface,* 7(47), 873-885.

[11] Vedula, S, and Rogers, P.P., 1981. Multi-objective analysis of irrigation planning in river basin development, Water Resources Research, 17: 1304-1310.

[12] Heffernan J. M., Smith R. J., and Wahl L.M., 2005. Perspectives on the basic reproductive ratio, *journal of the royal society of interface*. https://doi.org/10.1098/rsif.2005.0042.

[13] Driessche P. V., Watmough J. 2002. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission *Mathematical biosciences*, volume 180, Issues 1–2, Pages 29-48.