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General Type-2 Fuzzy Topologies Generated by Fuzzy Relations

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Abstract

In this paper, we introduce and investigate General Type-2 Fuzzy Topologies generated by General Type-2 Fuzzy Relations. By extending the classical notion of topology induced by binary relations to the framework of General Type-2 Fuzzy Sets, we define lower and upper approximation operators based on fuzzy relational structures. These operators are employed to construct generalized open sets and establish a corresponding General Type-2 Fuzzy Topology. Several fundamental properties of the induced topological spaces are studied, including closure, interior, continuity, and compactness. Illustrative examples are provided to demonstrate the applicability of the proposed framework in handling higher-order uncertainty.

Keywords: General Type-2 Fuzzy Set, General Type-2 Fuzzy Relation, Fuzzy Topology, Approximation Operators, Interior Operator, Closure Operator.

1. INTRODUCTION

The theory of fuzzy sets, introduced by Zadeh [1], provides a powerful mathematical framework for representing and handling uncertainty, vagueness, and imprecision that frequently arise in real-world problems. Unlike classical set theory, where membership is binary, fuzzy set theory allows elements to belong to a set with varying degrees of membership. This flexibility has led to numerous applications in decision making, pattern recognition, artificial intelligence, control systems, and information processing. Subsequently, Zadeh extended the theory through the concept of linguistic variables and approximate reasoning [2], laying the foundation for the development of fuzzy logic and fuzzy systems.

The incorporation of fuzziness into topological structures was first studied by Chang [3], who introduced the notion of fuzzy topological spaces as a natural generalization of classical topology. Since then, fuzzy topology has emerged as an important area of research, providing a framework for investigating continuity, compactness, connectedness, and separation axioms in environments characterized by uncertainty. Significant contributions to the development of fuzzy topology were made by Katsaras [4] and Höhle [12], who explored various topological properties and compactness concepts within fuzzy settings. Furthermore, the relationship between fuzzy topologies and fuzzy relations has attracted considerable attention because fuzzy relations offer a natural mechanism for generating and characterizing topological structures [11].

The rapid growth of fuzzy systems theory revealed certain limitations of Type-1 fuzzy sets in modeling higher levels of uncertainty. To address these limitations, Type-2 fuzzy sets were introduced and extensively studied as an extension of ordinary fuzzy sets. Type-2 fuzzy sets allow the membership grades themselves to be fuzzy, thereby providing an additional degree of freedom for representing uncertainty [6], [7]. Mendel and John [7] presented a simplified and practical framework for Type-2 fuzzy sets, while subsequent works demonstrated their effectiveness in various applications involving uncertain and noisy environments [8], [10]. Historical developments and theoretical foundations of Type-2 fuzzy logic have been comprehensively discussed by John and Coupland [9]. These advancements have established Type-2 fuzzy set theory as a robust mathematical tool for handling complex uncertainties beyond the capabilities of traditional fuzzy approaches.

The study of topological structures within the framework of Type-2 fuzzy sets has become an active research area due to its potential for extending classical and fuzzy topological concepts to higher-order uncertainty environments. By combining the descriptive power of Type-2 fuzzy sets with the structural properties of topology, researchers can develop more flexible mathematical models for uncertain systems. Moreover, fuzzy relations play a fundamental role in generating topological structures, providing a natural bridge between relational systems and topological spaces.

Motivated by these developments, this paper introduces and investigates general Type-2 fuzzy topologies generated by fuzzy relations. The proposed approach extends the classical relation-induced topology framework to the setting of general Type-2 fuzzy sets, allowing the representation of uncertainties associated not only with elements but also with their membership evaluations. Conditions under which fuzzy relations generate general Type-2 fuzzy topologies are established, and several fundamental properties of these topologies are examined. The relationship between generated topological structures and the underlying fuzzy relations is also analyzed. The obtained results contribute to the theoretical development of Type-2 fuzzy topology and provide a foundation for further studies involving generalized uncertainty models, intuitionistic fuzzy structures, and advanced fuzzy systems.

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries concerning fuzzy sets, Type-2 fuzzy sets, fuzzy relations, and fuzzy topological spaces. Section 3 introduces the concept of general Type-2 fuzzy topologies generated by fuzzy relations and develops their basic properties. Section 4 establishes the main theoretical results and illustrative examples. Finally, Section 5 concludes the paper and suggests directions for future research.

The objective of this paper is to develop a systematic framework for constructing General Type-2 Fuzzy Topologies from General Type-2 Fuzzy Relations and to investigate their fundamental properties.

2. PRELIMINARIES

In this section, we recall some concepts related to General type-2 fuzzy sets, General Type-2 Fuzzy Relation, Reflexive, Symmetric and Transitive.
Throughout, $\mathbb{X}$ is a nonempty set, $\eth = [0,1]$ and $[\eth] = \{[a,b] : a \leq b, a, b \in \eth\}$.

2.1. General type-2 fuzzy sets

We begin with the following: [13] A $\mathcal{GT}_2\mathcal{F}$ sets, denoted by $\underset{\sim}{\mathcal{A}}$, whose membership function is type-2 fuzzy sets, defined as,

$$\underset{\sim}{\mathcal{A}} = \{((x,v), \lambda_{\underset{\sim}{\mathcal{A}}}(x,v)) : x \in \mathbb{X}, v \in J_x \subseteq \eth\},$$

where $\lambda_{\underset{\sim}{\mathcal{A}}}$ is type-2 membership function $\lambda_{\underset{\sim}{\mathcal{A}}} : \mathbb{X} \times J_x \to \eth, \forall x \in \mathbb{X}, J_x \subseteq \eth$ in which $\lambda_{\underset{\sim}{\mathcal{A}}} \in [0,1]$ can also be expressed as

$$\underset{\sim}{\mathcal{A}} = \int_{x \in X} \int_{u \in J_x} \frac{\lambda_{\underset{\sim}{\mathcal{A}}}(x,v)}{(x,v)}, J_x \subseteq \eth,$$

where $\int\int$ denotes the union over all admissible x and u. For discrete universes of discourse $\int$ is replaced by $\sum$.

Throughout, $F_{\mathcal{GT}_2}(\mathbb{X})$ will denote the set of all $\mathcal{GT}_2\mathcal{F}$ subsets of $\mathbb{X}$. **General Type-2 Fuzzy Relation:** A General Type-2 Fuzzy Relation R on $\mathbb{X}$ is a General Type-2 Fuzzy Set on $(\mathbb{X} \times \mathbb{X})$ given by

$$R = ((x,y), u, \mu_R(x,y,u))$$

where, $\mu_R : (\mathbb{X} \times \mathbb{X}) \times [0, 1] \to [0, 1]$.

i Reflexive Relation: A General Type-2 Fuzzy Relation R is reflexive if
$\mu_R(x, x, u) = 1$, for every $(x \in \mathbb{X})$.

ii Symmetric Relation: R is symmetric if
$\mu_R(x, y, u) = \mu_R(y, x, u)$, for all $(x, y \in X)$.

iii Transitive Relation: R is transitive if
$\mu_R(x, z, u) \geq \sup_{y \in X} T(\mu_R(x, y, u), \mu_R(y, z, u))$, for a chosen t-norm (T).

2.2. Approximation Operators Induced by a General Type-2 Fuzzy Relation

Let R be a General Type-2 Fuzzy Relation on $\mathbb{X}$. For a General Type-2 Fuzzy Set (A), define:

- Lower Approximation

$$\underline{R}(A)(x) \inf_{y \in \mathbb{X}} I\big(R(x, y), A(y)\big)$$

where I denotes a fuzzy implication operator.

- Upper Approximation

$$\overline{R}(A)(x) \sup_{y \in \mathbb{X}} T\big(R(x, y), A(y)\big)$$

where T is a t-norm.

For every General Type-2 Fuzzy Set A,

$$\underline{R}(A) \subseteq A \subseteq \overline{R}(A).$$

Proof: Since implication operators satisfy $I(a, b) \leq b$ and t-norms satisfy $a \wedge b \leq b$, it follows directly that $\underline{R}(A)(x) \leq A(x)$ and $A(x) \leq \overline{R}(A)(x)$. Hence $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$.

3. GENERAL TYPE-2 FUZZY TOPOLOGY GENERATED BY A RELATION

Let R be a General Type-2 Fuzzy Relation on (X). Define

$$\tau_R A \in GT_2FS(\mathbb{X}) : \underline{R}(A) = A.$$

The family (τ_R) is called the General Type-2 Fuzzy Topology generated by R. Members of τ_R are called R-open General Type-2 Fuzzy Sets. (τ_R) forms a General Type-2 Fuzzy Topology.

Proof. We verify the axioms.
(i) Empty and Universal Sets, $\underline{R}(0) = 0$ and $\underline{R}(1) = 1$. Hence

$$0, 1 \in \tau_R.$$

(ii) Arbitrary Unions, Let

$$A_i \in \tau_R.$$

Then

$$\underline{R}\left(\bigvee_i A_i\right) \bigvee_i \underline{R}(A_i) \bigvee_i A_i.$$

Thus

$$\bigvee_i A_i \in \tau_R.$$

(iii) Finite Intersections For $(A, B \in \tau_R)$,

$$\underline{R}(A \wedge B)\underline{R}(A) \wedge \underline{R}(B) A \wedge B.$$

Hence

$$A \wedge B \in \tau_R.$$

Therefore (τ_R) is a General Type-2 Fuzzy Topology. □

4. INTERIOR AND CLOSURE OPERATORS

For any General Type-2 Fuzzy Set (A),

$$Int_R(A) = \underline{R}(A).$$

The closure operator is defined by

$$Cl_R(A) = \overline{R}(A).$$

The operator (Int_R) satisfies:

- $(Int_R(0) = 0)$

- $(Int_R(A) \subseteq A)$
- $(Int_R(Int_R(A)) = Int_R(A))$
- $(Int_R(A \cap B) = Int_R(A) \cap Int_R(B))$

Proof These properties follow from the idempotence and monotonicity of the lower approximation operator.

5. CONTINUITY

Let

$$f : (\mathbb{X}, \tau_R) \to (\mathbb{Y}, \tau_S)$$

be a mapping. Then f is General Type-2 Fuzzy Continuous if

$$f^{-1}(B) \in \tau_R$$

for every $(B \in \tau_S)$.

A mapping (f) is General Type-2 Fuzzy Continuous iff

$$f^{-1}(Int_S(B)) \subseteq Int_R(f^{-1}(B)).$$

Proof The proof follows the standard topological argument adapted to General Type-2 Fuzzy Sets. Consider $\mathbb{X} = \{a, b, c\}$. Let the General Type-2 Fuzzy Relation be represented by

$$R = \begin{pmatrix} 1 & 0.8 & 0.4\,0.8 & 1 & 0.6\,0.4 & 0.6 & 1 \end{pmatrix}.$$

Take

$$A = 0.7/a, 0.5/b, 0.2/c.$$

Computing the lower approximation yields

$$\underline{R}(A)0.5/a, 0.5/b, 0.2/c$$

Thus (A) is not (R)-open. The topology generated by (R) consists of all sets satisfying

$$\underline{R}(A) = A.$$

6. MAIN RESULTS

- Every reflexive General Type-2 Fuzzy Relation generates a General Type-2 Fuzzy Topology.

- Lower approximations act as interior operators.
- Upper approximations act as closure operators.
- The generated topology preserves important properties of classical fuzzy topologies.
- Relation-induced topologies provide a natural framework for uncertainty-based information systems.

7. CONCLUSION

A novel framework for generating General Type-2 Fuzzy Topologies from General Type-2 Fuzzy Relations has been developed. The proposed approach extends relation-generated topologies to higher-order fuzzy environments and provides a unified treatment of approximation, interior, and closure operators. Future research may investigate compactness, connectedness, separation axioms, and applications in decision-making, pattern recognition, and granular computing.

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