Intuitionistic fuzzy $\ell$–subsemiring of a $\ell$–semiring $R$
under homomorphism and anti-homomorphism

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Abstract

In this paper, we made an attempt to study the properties of intuitionistic fuzzy $\ell$–subsemiring of a $\ell$–semiring under homomorphism and anti-homomorphism and we introduce some theorems on this.

Key Words: Fuzzy subset, intuitionistic fuzzy $\ell$–subsemiring, homomorphism, anti-homomorphism and strongest intuitionistic fuzzy relation.

Introduction

The concept of fuzzy sets was initiated by L.A.Zadeh [7] in 1965. After the introduction of fuzzy sets several researchers explored on the generalization of the concept of fuzzy sets. K.T.Atanassov introduced [1] intuitionistic fuzzy subset in 1983. George Gargor named new sets as the intuitionistic fuzzy subset. In this paper to introduced the concept of intuitionistic fuzzy $\ell$–subsemiring of a $\ell$–semiring under homomorphism and anti-homomorphism and established some results on these.

Definition: 1.1

Let $R$ be a $\ell$–semiring. An intuitionistic fuzzy subset $A$ of $R$ is said to be an intuitionistic fuzzy $\ell$–subsemiring of $R$ if it satisfies the following conditions:

(i) $\mu_A(x + y) \geq \min \{\mu_A(x) , \mu_A(y)\}$
(ii) $\mu_A(xy) \geq \min \{\mu_A(x) , \mu_A(y)\}$
(iii) $\mu_A(x \lor y) \geq \min \{\mu_A(x) , \mu_A(y)\}$
(iv) $\mu_A(x \land y) \geq \min \{\mu_A(x) , \mu_A(y)\}$
\( v_A(x+y) \leq \max \{v_A(x), v_A(y)\} \)

\( v_A(x \cdot y) \leq \max \{v_A(x), v_A(y)\} \)

\( v_A(x \lor y) \leq \max \{v_A(x), v_A(y)\} \)

\( v_A(x \land y) \leq \max \{v_A(x), v_A(y)\} \)

for all \( x, y \in R \)

**Definition: 1.2**

Let \( A \) be an intuitionistic fuzzy subset in a set \( S \), the strongest intuitionistic fuzzy relation on \( S \), that is a intuitionistic fuzzy relation on \( A \) is \( V \) given by

\[ \mu_V(x,y) = \min \{\mu_A(x), \mu_A(y)\} \]

\[ v_V(x,y) = \max \{v_A(x), v_A(y)\} \]

for all \( x, y \in S \).

**Definition: 1.3**

Let \( R \) and \( R' \) be any two \( \ell \)-semirings. Then the function \( f : R \rightarrow R' \) is called a \( \ell \)-semiring homomorphism if it satisfies the following axioms:

(i) \( f(x+y) = f(x) + f(y) \)

(ii) \( f(xy) = f(x)f(y) \)

(iii) \( f(x \lor y) = f(x) \lor f(y) \)

(iv) \( f(x \land y) = f(x) \land f(y) \),

for all \( x, y \in R \)

**Example: 1.1**

Let \( R = \{m+n\sqrt{2} \mid m, n \in \mathbb{Z}\} \). \( R \) is a \( \ell \)-semiring under usual addition and multiplication

Define \( f : R \rightarrow R \) by \( f(m+n\sqrt{2}) = m-n\sqrt{2} \) is \( \ell \)-semiring homomorphism, where \( \mathbb{Z} \) is the set of all integers.

**Definition: 1.4**

Let \( R \) and \( R' \) be any two \( \ell \)-semirings. Then the function \( f : R \rightarrow R' \) is called a \( \ell \)-semiring anti-homomorphism if it satisfies the following axioms:

(i) \( f(x+y) = f(y) + f(x) \)

(ii) \( f(xy) = f(y)f(x) \)

(iii) \( f(x \lor y) = f(y) \lor f(x) \)

(iv) \( f(x \land y) = f(y) \land f(x) \),

for all \( x, y \in R \)
Theorem: 1.1
Let $R$ and $R'$ be any two $\ell$−semirings. The homomorphic image of an intuitionistic fuzzy $\ell$−subsemiring of $R$ is an intuitionistic fuzzy $\ell$−subsemiring of $R'$.

Proof:
Let $R$ and $R'$ be any two $\ell$−semirings and $f : R \to R'$ an homomorphism.
Let $V = f(A)$, where $A$ is an intuitionistic fuzzy $\ell$−subsemiring of $R$.
To prove $V$ is an intuitionistic fuzzy $\ell$−subsemiring of $R'$.
For $f(x)$, $f(y)$ in $R'$,
(i) $\mu_V(f(x) + f(y)) = \mu_V(f(x + y)) \geq \mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$
$\Rightarrow \mu_V(f(x) + f(y)) \geq \min\{\mu_V(x), \mu_V(y)\}$, for all $x, y$ in $R$
(ii) $\mu_V(f(x)f(y)) = \mu_V(f(xy)) \geq \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$
$\Rightarrow \mu_V(f(x)f(y)) \geq \min\{\mu_V(x), \mu_V(y)\}$, for all $x, y$ in $R$
(iii) $\mu_V(f(x) \vee f(y)) = \mu_V(f(x \vee y)) \geq \mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$
$\Rightarrow \mu_V(f(x) \vee f(y)) \geq \min\{\mu_V(x), \mu_V(y)\}$, for all $x, y$ in $R$
(iv) $\mu_V(f(x) \wedge f(y)) = \mu_V(f(x \wedge y)) \geq \mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$
$\Rightarrow \mu_V(f(x) \wedge f(y)) \geq \min\{\mu_V(x), \mu_V(y)\}$, for all $x, y$ in $R$
(v) $v_V(f(x) + f(y)) = v_V(f(x + y)) \leq v_A(x + y) \leq \max\{v_A(x), v_A(y)\}$
$\Rightarrow v_V(f(x) + f(y)) \leq \max\{v_V(x), v_V(y)\}$, for all $x, y$ in $R$
(vi) $v_V(f(x)f(y)) = v_V(f(xy)) \leq v_A(xy) \leq \max\{v_A(x), v_A(y)\}$
$\Rightarrow v_V(f(x)f(y)) \leq \max\{v_V(x), v_V(y)\}$, for all $x, y$ in $R$
(vii) $v_V(f(x) \vee f(y)) = v_V(f(x \vee y)) \leq v_A(x \vee y) \leq \max\{v_A(x), v_A(y)\}$
$\Rightarrow v_V(f(x) \vee f(y)) \leq \max\{v_V(x), v_V(y)\}$, for all $x, y$ in $R$
(viii) $v_V(f(x) \wedge f(y)) = v_V(f(x \wedge y)) \leq v_A(x \wedge y) \leq \max\{v_A(x), v_A(y)\}$
$\Rightarrow v_V(f(x) \wedge f(y)) \leq \max\{v_V(x), v_V(y)\}$, for all $x, y$ in $R$
Therefore $V$ is an intuitionistic fuzzy $\ell$−subsemiring of $R'$.

Theorem: 1.2
Let $R$ and $R'$ be any two $\ell$−semirings. The homomorphic pre-image of an intuitionistic fuzzy $\ell$−subsemiring of $R'$ is an intuitionistic fuzzy $\ell$−subsemiring of $R$. 


Proof:

Let $R$ and $R'$ be any two $\ell-$semirings and $f : R \rightarrow R'$ a homomorphism.

Let $V = f(A)$, where $V$ is an intuitionistic fuzzy $\ell-$subsemiring of $R'$.

To prove $A$ is an intuitionistic fuzzy $\ell-$subsemiring of $R$. For $x, y$ in $R$,

(i) $\mu_A(x + y) = \mu_V(f(x + y))$, since $\mu_V(f(x)) = \mu_A(x)$

$= \mu_V(f(x) + f(y))$, ($\because f$ is a homomorphism)

$\geq \min \{\mu_V(f(x)), \mu_V(f(y))\}$

$= \min \{\mu_A(x), \mu_A(y)\}$, since $\mu_V(f(x)) = \mu_A(x)$

$\Rightarrow \quad \mu_A(x + y) \geq \min \{\mu_A(x), \mu_A(y)\}$, for all $x, y$ in $R$

(ii) $\mu_A(xy) = \mu_V(f(xy))$, since $\mu_V(f(x)) = \mu_A(x)$

$= \mu_V(f(x)f(y))$, ($\because f$ is a homomorphism)

$\geq \min \{\mu_V(f(x)), \mu_V(f(y))\}$

$= \min \{\mu_A(x), \mu_A(y)\}$, since $\mu_V(f(x)) = \mu_A(x)$

$\Rightarrow \quad \mu_A(xy) \geq \min \{\mu_A(x), \mu_A(y)\}$, for all $x, y$ in $R$

(iii) $\mu_A(x \vee y) = \mu_V(f(x \vee y))$, since $\mu_V(f(x)) = \mu_A(x)$

$= \mu_V(f(x) \vee f(y))$, ($\because f$ is a homomorphism)

$\geq \min \{\mu_V(f(x)), \mu_V(f(y))\}$

$= \min \{\mu_A(x), \mu_A(y)\}$, since $\mu_V(f(x)) = \mu_A(x)$

$\Rightarrow \quad \mu_A(x \vee y) \geq \min \{\mu_A(x), \mu_A(y)\}$, for all $x, y$ in $R$

(iv) $\mu_A(x \wedge y) = \mu_V(f(x \wedge y))$, since $\mu_V(f(x)) = \mu_A(x)$

$= \mu_V(f(x) \wedge f(y))$, ($\because f$ is a homomorphism)

$\geq \min \{\mu_V(f(x)), \mu_V(f(y))\}$

$= \min \{\mu_A(x), \mu_A(y)\}$, since $\mu_V(f(x)) = \mu_A(x)$

$\Rightarrow \quad \mu_A(x \wedge y) \geq \min \{\mu_A(x), \mu_A(y)\}$, for all $x, y$ in $R$

(v) $v_A(x + y) = v_V(f(x + y))$, since $v_V(f(x)) = v_A(x)$

$= v_V(f(x) + f(y))$, ($\because f$ is a homomorphism)

$\leq \max \{v_V(f(x)), v_V(f(y))\}$

$= \max \{v_A(x), v_A(y)\}$, since $\mu_V(f(x)) = \mu_A(x)$
\[ \Rightarrow v_A(x + y) \leq \max \{v_A(x), v_A(y)\}, \text{ for all } x, y \in R \]

(vi) \[ v_A(xy) = v_V(f(xy)), \text{ since } v_V(f(x)) = v_A(x) \]
\[ = v_V(f(x)f(y)), (\because f \text{ is a homomorphism}) \]
\[ \leq \max \{v_V(f(x)), v_V(f(y))\} \]
\[ = \max \{v_A(x), v_A(y)\}, \text{ since } \mu_V(f(x)) = \mu_A(x) \]

\[ \Rightarrow v_A(xy) \leq \max \{v_A(x), v_A(y)\}, \text{ for all } x, y \in R \]

(vii) \[ v_A(x \vee y) = v_V(f(x \vee y)), \text{ since } v_V(f(x)) = v_A(x) \]
\[ = v_V(f(x) \vee f(y)), (\because f \text{ is a homomorphism}) \]
\[ \leq \max \{v_V(f(x)), v_V(f(y))\} \]
\[ = \max \{v_A(x), v_A(y)\}, \text{ since } \mu_V(f(x)) = \mu_A(x) \]

\[ \Rightarrow v_A(x \vee y) \leq \max \{v_A(x), v_A(y)\}, \text{ for all } x, y \in R \]

Therefore \( A \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R \).

**Theorem: 1.3**

Let \( R \) and \( R' \) be any two \( \ell - \) semirings. The anti-homomorphic image of an intuitionistic fuzzy \( \ell - \) subsemiring of \( R \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R' \).

**Proof:**

Let \( R \) and \( R' \) be any two \( \ell - \) semirings and \( f : R \to R' \) an anti-homomorphism.

Let \( V = f(A) \), where \( A \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R \).

To prove \( V \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R' \).

For \( f(x), f(y) \) in \( R' \),

(i) \[ \mu_V(f(x) + f(y)) = \mu_V(f(y + x)), \text{ since } f \text{ is an anti-homomorphism} \]
\[ \geq \mu_A(y + x) \geq \min \{\mu_A(y), \mu_A(x)\} = \min \{\mu_A(x), \mu_A(y)\} \]
\[ \Rightarrow \mu_V(f(x) + f(y)) \geq \min \{\mu_V(f(x)), \mu_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

(ii) \[ \mu_V(f(x)f(y)) = \mu_V(f(yx)), \text{ since } f \text{ is an anti-homomorphism} \]
\[ \geq \mu_A(yx) \geq \min \{\mu_A(y), \mu_A(x)\} = \min \{\mu_A(x), \mu_A(y)\} \]

\[ \Rightarrow \mu_V(f(x)f(y)) \geq \min \{\mu_V(f(x)), \mu_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

(iii) \[ \mu_V(f(x)f(y)) = \mu_V(f(yf(x)) \geq \min \{\mu_A(y), \mu_A(x)\} = \min \{\mu_A(x), \mu_A(y)\} \]

\[ \Rightarrow \mu_V(f(x)f(y)) \geq \min \{\mu_V(f(x)), \mu_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

(iv) \[ \mu_V(f(x)f(y)) = \mu_V(f(yf(x)), \text{ since } f \text{ is an anti-homomorphism} \]

\[ \geq \mu_A(yf(x) \geq \min \{\mu_A(x), \mu_A(y)\} \]

\[ \Rightarrow \mu_V(f(x)f(y)) \geq \min \{\mu_V(f(x)), \mu_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

(v) \[ v_V(f(x)+f(y)) = v_V(f(y+x)), \text{ since } f \text{ is an anti-homomorphism} \]

\[ \leq v_A(x+y) \leq \max \{v_A(x), v_A(y)\} \]

\[ \Rightarrow v_V(f(x)+f(y)) \leq \max \{v_V(f(x)), v_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

(vi) \[ v_V(f(x)f(y)) = v_V(f(yx)), \text{ since } f \text{ is an anti-homomorphism} \]

\[ \leq v_A(x+y) \leq \max \{v_A(x), v_A(y)\} \]

\[ \Rightarrow v_V(f(x)f(y)) \leq \max \{v_V(f(x)), v_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

(vii) \[ v_V(f(x)f(y)) = v_V(f(yx)), \text{ since } f \text{ is an anti-homomorphism} \]

\[ \leq v_A(x+y) \leq \max \{v_A(x), v_A(y)\} \]

\[ \Rightarrow v_V(f(x)f(y)) \leq \max \{v_V(f(x)), v_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

(viii) \[ v_V(f(x)f(y)) = v_V(f(yx)), \text{ since } f \text{ is an anti-homomorphism} \]

\[ \leq v_A(x+y) \leq \max \{v_A(x), v_A(y)\} \]

\[ \Rightarrow v_V(f(x)f(y)) \leq \max \{v_V(f(x)), v_V(f(y))\}, \text{ for all } x, y \text{ in } R \]

Therefore \( V \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R' \).

**Theorem: 1.4**

Let \( R \) and \( R' \) be any two \( \ell - \) semirings. The anti-homomorphic pre-image of an intuitionistic fuzzy \( \ell - \) subsemiring of \( R' \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R \).

**Proof:**

Let \( R \) and \( R' \) be any two \( \ell - \) semirings and \( f : R \rightarrow R' \) an anti-homomorphism.

Let \( V = f(A) \), where \( V \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R' \).

To prove \( A \) is an intuitionistic fuzzy \( \ell - \) subsemiring of \( R \). For \( x, y \in R \),
(i) \( \mu_A(x+y) = \mu_V(f(x+y)) \), since \( \mu_V(f(x)) = \mu_A(x) \)
\( = \mu_V(f(y)+f(x)), (\because f \text{ is an anti- homomorphism}) \)
\( \geq \min \{\mu_V(f(y)), \mu_V(f(x))\} = \min \{\mu_V(f(x)), \mu_V(f(y))\} \)
\( = \min \{\mu_A(x), \mu_A(y)\}, \text{ (since } \mu_V(f(x)) = \mu_A(x)\) \)
\( \Rightarrow \mu_A(x+y) \geq \min \{\mu_A(x), \mu_A(y)\}, \text{ for all } x, y \in R \)

(ii) \( \mu_A(xy) = \mu_V(f(x)\cdot y) \), since \( \mu_V(f(x)) = \mu_A(x) \)
\( = \mu_V(f(y)f(x)), (\because f \text{ is an anti- homomorphism}) \)
\( \geq \min \{\mu_V(f(y)), \mu_V(f(x))\} = \min \{\mu_V(f(x)), \mu_V(f(y))\} \)
\( = \min \{\mu_A(x), \mu_A(y)\}, \text{ since } \mu_V(f(x)) = \mu_A(x) \)
\( \Rightarrow \mu_A(xy) \geq \min \{\mu_A(x), \mu_A(y)\}, \text{ for all } x, y \in R \)

(iii) \( \mu_A(x \lor y) = \mu_V(f(x \lor y)) \), since \( \mu_V(f(x)) = \mu_A(x) \)
\( = \mu_V(f(y) \lor f(x)), (\because f \text{ is an anti- homomorphism}) \)
\( \geq \min \{\mu_V(f(y)), \mu_V(f(x))\} = \min \{\mu_V(f(x)), \mu_V(f(y))\} \)
\( = \min \{\mu_A(x), \mu_A(y)\}, \text{ since } \mu_V(f(x)) = \mu_A(x) \)
\( \Rightarrow \mu_A(x \lor y) \geq \min \{\mu_A(x), \mu_A(y)\}, \text{ for all } x, y \in R \)

(iv) \( \mu_A(x \land y) = \mu_V(f(x \land y)) \), since \( \mu_V(f(x)) = \mu_A(x) \)
\( = \mu_V(f(y) \land f(x)), (\because f \text{ is an anti- homomorphism}) \)
\( \geq \min \{\mu_V(f(y)), \mu_V(f(x))\} = \min \{\mu_V(f(x)), \mu_V(f(y))\} \)
\( = \min \{\mu_A(x), \mu_A(y)\}, \text{ since } \mu_V(f(x)) = \mu_A(x) \)
\( \Rightarrow \mu_A(x \land y) \geq \min \{\mu_A(x), \mu_A(y)\}, \text{ for all } x, y \in R \)

(v) \( \nu_A(x+y) = \nu_V(f(x+y)) \), since \( \nu_V(f(x)) = \nu_A(x) \)
\( = \nu_V(f(y)+f(x)), (\because f \text{ is an anti- homomorphism}) \)
\( \leq \max \{\nu_V(f(y)), \nu_V(f(x))\} = \max \{\nu_V(f(x)), \nu_V(f(y))\} \)
\( = \max \{\nu_A(x), \nu_A(y)\}, \text{ since } \nu_V(f(x)) = \nu_A(x) \)
\( \Rightarrow \nu_A(x+y) \leq \max \{\nu_A(x), \nu_A(y)\}, \text{ for all } x, y \in R \)

(vi) \( \nu_A(xy) = \nu_V(f(xy)) \), since \( \nu_V(f(x)) = \nu_A(x) \)
\( = \nu_V(f(y)f(x)), (\because f \text{ is an anti- homomorphism}) \)
\( \leq \max \{\nu_V(f(y)), \nu_V(f(x))\} = \max \{\nu_V(f(x)), \nu_V(f(y))\} \)
\[ v_A(x \land y) = v_V(f(x \land y)), \text{ since } v_V(f(x))=v_A(x) \]
\[ = v_V(f(y) \land f(x)), (\because f \text{ is an anti-homomorphism}) \]
\[ \leq \max \{v_V(f(y)), v_V(f(x))\} = \max \{v_V(f(x)), v_V(f(y))\} \]
\[ = \max \{v_A(x), v_A(y)\}, \text{ since } v_V(f(x))=v_A(x) \]
\[ \Rightarrow v_A(x \land y) \leq \max \{v_A(x), v_A(y)\}, \text{ for all } x, y \in R \]

(vii) \[ v_A(x \lor y) = v_V(f(x \lor y)), \text{ since } v_V(f(x))=v_A(x) \]
\[ = v_V(f(y) \lor f(x)), (\because f \text{ is an anti-homomorphism}) \]
\[ \leq \max \{v_V(f(y)), v_V(f(x))\} = \max \{v_V(f(x)), v_V(f(y))\} \]
\[ = \max \{v_A(x), v_A(y)\}, \text{ since } v_V(f(x))=v_A(x) \]
\[ \Rightarrow v_A(x \lor y) \leq \max \{v_A(x), v_A(y)\}, \text{ for all } x, y \in R \]

Therefore \( A \) is an intuitionistic fuzzy \( \ell \)–subsemiring of \( R \).

References: