Fuzzy Closure Operator Induced by a Fuzzy Pseudo Metric

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Abstract

In this paper, we define fuzzy pseudo metric space and show that to every fuzzy pseudo metric, a fuzzy Čech closure operator is associated and prove that the fuzzy closure operator induced by a fuzzy pseudo metric is fuzzy topological.

AMS subject classification:

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1. Introduction

In 1965, L. A. Zadeh [6] introduced the concept of fuzzy sets. Fuzzy sets parallel in many respects the ordinary sets, but are more general and have a wider scope of applicability. In 1968, C.L. Chang [3] applied the fuzzy set theory to topology and introduced the fuzzy topology. He used the unit closed interval [0,1] as the membership set. Since then an extensive study of fuzzy topological spaces has been carried out by many mathematicians and some of them used a complete lattice as the membership set instead of [0,1] and hence generalized the concept of fuzzy topology to L- fuzzy topology. In 1965, Čech E. introduced the concept of a Čech closure operator and defined Čech closure spaces. We apply fuzzy set theory to Čech closure operator and introduces the concept of fuzzy Čech closure operator. Corresponding to each fuzzy Čech closure operator V, there is a fuzzy topology called the associated fuzzy topology of V. But there can have two different fuzzy Čech closure operators on a set with same associated fuzzy topology. In this sense fuzzy Čech closure spaces can be considered as generalization of fuzzy topological spaces. In this paper, we introduce the concept of a fuzzy pseudo metric and shows that each fuzzy Čech closure operator induced by a fuzzy pseudo metric is fuzzy topological.
Definition 1.1. Let $X$ be a nonempty set, $L$ a complete lattice. An $L$-fuzzy subset $A$ of $X$ is a mapping $A : X \rightarrow L$. The family of all $L$-fuzzy subsets of $X$ is denoted by $L^X$. For brevity, we call an $L$-fuzzy subset of $X$ as a fuzzy subset of $X$.

$L^X$ is a complete lattice with the partial order “$\leq$” defined by: For all $A, B \in L^X$, $A \leq B \iff A(x) \leq B(x)$ for all $x \in X$.

Definition 1.2. A fuzzy point on $X$ is a fuzzy subset $x_l$ in $L^X$ defined as, for all $y \in X$

$$x_l(y) = \begin{cases} l & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

where $x \in X$ and $l$ is a non-zero element of $L$.

Definition 1.3. Let $L$ be a lattice. An element $l \in L$ is called a molecule if $l > 0$ and for all $a, b \in L$, $l = a \lor b \implies l = a$ or $l = b$. The set of all molecules in $L$ is denoted by $M(L)$.

Let $X$ be a nonempty set and $L$ be a complete, distributive and pseudo-complemented lattice with $M(L) = L - \{0\}$. Then the set of all fuzzy points in $L^X$ are molecules of $L^X$ and we denote this set by $D$.

Definition 1.4. Let $d : D \times D \rightarrow [0, \infty)$ satisfying the conditions:

(i) $d(x_l, x_m) = 0$ where $l \leq m$.

(ii) $d(x_l, y_m) = d(y_m, x_l)$

(iii) $d(x_l, z_n) \leq d(x_l, y_m) + d(y_m, z_n)$

(iv) if $d(x_l, y_m) < r$ where $r > 0$, then there exists $n > l$ such that $d(x_n, y_m) < r$,

where $x, y, z \in X$ and $l, m, n \in L - \{0\}$.

Then $d$ is called a fuzzy pseudo metric on $X$ and $(X, d)$ is a fuzzy pseudo metric space.

Definition 1.5. Let $Q^r_{x_a} = \{y_b : d(y_b, x_a) < r\}$ and $B(x_a, r) = \bigvee\{y_b : y_b \in Q^r_{x_a}\}$. Then $B(x_a, r)$ is called a sphere with center $x_a$ and radius $r$.

Definition 1.6. $A \in L^X$ is said to be open if and only if for every $x_a \in A$, there exists $B(x_a, r) \leq A$.

Definition 1.7. Let $X$ be a nonempty set and $L$ be an $F$-lattice. A fuzzy Čech closure operator on $X$ is a mapping $\psi : L^X \rightarrow L^X$ satisfying the conditions:

(i) $\psi(\emptyset) = 0$

(ii) $A \leq \psi(A)$ for all $A \in L^X$
(iii) $\psi(A \lor B) = \psi(A) \lor \psi(B)$ for all $A, B \in L^X$.

For brevity, we call $\psi$ a fuzzy closure operator on $X$. The pair $(L^X, \psi)$ or $(X, \psi)$ is called a fuzzy closure space.

**Definition 1.8.** Let $(X, d)$ be a fuzzy pseudo metric space. The number $d(x_l, y_m)$ is called the distance from $x_l$ to $y_m$ in $(X, d)$ under $d$. If $A, B \in L^X$, then $d(A, B)$ is defined to be

$$d(A, B) = \begin{cases} \infty & \text{if } A = 0 \text{ or } B = 0, \\ \inf \{d(x_l, y_m) : x_l \in A \text{ and } y_m \in B\} & \text{otherwise.} \end{cases}$$

The distance $d(x_l, A)$ from a fuzzy point $x_l$ to a fuzzy set $A$ is defined to be $\inf \{d(x_l, y_m) : y_m \in A\}$. A mapping $f : (X, d) \rightarrow (X_1, d_1)$ is said to be distance preserving, if $d_1(f(x_l), f(y_m)) = d(x_l, y_m)$ for each $x_l, y_m \in D$. Two fuzzy pseudo metric spaces are said to be isomorphic, if there exists a distance preserving bijective mapping of one onto the other. With every fuzzy pseudo metric there is associated a fuzzy closure operator which will be described as follows: Let $d$ be a fuzzy pseudo metric for a set $X$. For $A \in L^X$, the relation

$$\psi(A) = \begin{cases} A & \text{if } A = 0, \\ \bigvee \{x_l : d(x_l, A) = 0\} & \text{otherwise.} \end{cases} \quad (\ast)$$

is a fuzzy closure operator on $X$, for, obviously $\psi$ is a mapping from $L^X$ into $L^X$ and,

(i) $\psi(0) = 0$

(ii) $x_l \in A \Rightarrow d(x_l, A) = 0$ yields $A \leq \psi(A)$

(iii) since $d(x_l, A \lor B) = \min\{d(x_l, A), d(x_l, B)\}$, we get $\psi(A \lor B) = \psi(A) \lor \psi(B)$.

**Definition 1.9.** If $d$ is a fuzzy pseudo metric on a set $X$, then the fuzzy closure operator $\psi$ defined by $(\ast)$ is said to be the fuzzy closure operator induced by $d$.

**Remark 1.10.** Every fuzzy pseudo metric space $(X, d)$ will be considered as a fuzzy closure space $(X, \psi)$ where $\psi$ is the fuzzy closure operator induced by $d$. For example, if we say that $f$ is a continuous mapping of a fuzzy pseudo metric space $(X, d_1)$ into another one $(X, d_2)$, it is to be understood that the mapping $f : (X, \psi_1) \rightarrow (X, \psi_2)$ is continuous where $\psi_i$ is the fuzzy closure operators induced by $d_i$. Similarly we shall speak about closed or open subsets of a fuzzy pseudo metric space.

**Definition 1.11.** A fuzzy closure operator $\psi$ (or a fuzzy closure space $(X, \psi)$) is said to be fuzzy pseudo metrizable, if $\psi$ is induced by a fuzzy pseudo metric. For convenience, two pseudo metrics on the same set will be called fuzzy topologically equivalent, if they induce the same fuzzy closure operators.
Definition 1.12. Let \((X, \psi)\) and \((Y, \phi)\) be fuzzy closure spaces. Then the mapping \(f : (X, \psi) \rightarrow (Y, \phi)\) is said to be continuous if and only if the inverse image of each open (closed) subset of \((Y, \phi)\) is an open (closed) subset of \((X, \psi)\).

Example 1.13. Let \(X\) be a set. The relation \(d : (x_l, y_m) \rightarrow 0\) for every \((x_l, y_m) \in D \times D\) is a fuzzy pseudo metric on \(D\) inducing the indiscrete fuzzy closure operator on \(X\). Conversely, if a fuzzy pseudo metric induces the indiscrete fuzzy closure operator on \(X\), then necessarily \(d(x_l, y_m) = 0\) for each \((x_l, y_m) \in D \times D\) because \(d(x_l, y_m) \neq 0\) implies \(x_l\) not belongs to the closure of \(y_m\).

Example 1.14. Given a set \(X\), consider \(d\) which assigns to a pair \((x_l, y_m)\), the element 0 if \(x_l = y_m\) and 1 if \(x_l \neq y_m\). Then \(d\) is a fuzzy pseudo metric on \(X\) inducing the indiscrete fuzzy closure operator on \(X\).

Example 1.15. If \(d\) is a fuzzy pseudo metric on the set \(D\) and \(r\) is a positive real, then the relation \((x_l, y_m) \rightarrow r.d(x_l, y_m)\) denoted by \(r.d\) is a fuzzy pseudo metric on \(X\) inducing the same fuzzy closure operator as \(d\). Since clearly \(d = r.d\) if and only if \(d(x_l, y_m) = 0\) for each \((x_l, y_m)\), we obtain that the indiscrete fuzzy closure operator on \(X\) is the only fuzzy closure operator on \(X\) induced by exactly one fuzzy pseudo metric.

2. Main Results

Proposition 2.1. A mapping \(f\) of a fuzzy pseudo metric space \((X, d)\) into another one \((X_1, d_1)\) is continuous at a fuzzy point \(x_l\) of \(L^X\) if and only if the following conditions is fulfilled: For each positive real \(r\), there exists a positive real \(s\) such that \(d(x_l, y_m) < s \implies d_1(f(x_l), f(y_m)) < r\).

Proof. The implication \(d(x_l, y_m) < s \implies d_1(f(x_l), f(y_m)) < r\) is equivalent to this assertion: the image under \(f\) of the open \(s\)-sphere about \(x_l\) in \((X, d)\) is contained in the open \(r\)-sphere about \(f(x_l)\) in \((X_1, d_1)\). Since open spheres form local bases, the statement follows. \(\blacksquare\)

Definition 2.2. A Lipschitz continuous mapping or simply a Lipschitz mapping of a fuzzy pseudo metric space \((X, d)\) into another one \((X_1, d_1)\) is a mapping \(f : (X, d) \Rightarrow (X_1, d_1)\) such that there exists a non-negative \(K\), called a Lipschitz bound of \(f\), with \(K.d(x_l, y_m) \geq d_1(f(x_l), f(y_m))\) for each \((x_l, y_m) \in D \times D\).

Proposition 2.3. Every Lipschitz continuous mapping is continuous.

Proof. Let \(f\) be a Lipschitz continuous mapping of \((X, d)\) into \((X_1, d_1)\) and let always \(K.d(x_l, y_m) \geq d_1(f(x_l), f(y_m))\) where \(K\) is a positive real. Given \(r > 0\) put \(s = r.K^{-1}\) and apply Preposition 2.1. \(\blacksquare\)

Proposition 2.4. If \((X, d)\) is a fuzzy pseudo metric space, \(x_l, y_m, z_n, t_p\) are fuzzy points of \(L^X\) and \(A\) is a nonzero fuzzy subset of \(L^X\), then
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(i) \(|d(x_l, A) - d(y_m, A)| \leq d(x_l, y_m)\).

(ii) \(|d(x_l, y_m) - d(z_n, t_p)| \leq d(x_l, z_n) + d(y_m, t_p)\).

Proof.

(i) If \(z_n \in A\), then \(d(x_l, A) \leq d(x_l, z_n)\) and by the triangle inequality \(d(x_l, A) \leq d(x_l, y_m) + d(y_m, z_n)\). Taking the g.l.b. of \(d(y_m, z_n)\) for \(z_n \in A\), we obtain \(d(x_l, A) \leq d(x_l, y_m) + d(y_m, z_n)\). The same inequality holds with \(x_l\) and \(y_m\) interchanged.

(ii) Formula (ii) follows by a double application of the triangle inequality:

\[d(x_l, y_m) \leq d(x_l, z_n) + d(z_n, t_p) + d(t_p, y_m)\]

and consequently \(d(x_l, y_m) - d(z_n, t_p) \leq d(x_l, z_n) + d(y_m, t_p)\). Now the conclusion follows as in the proof of (a).

Definition 2.5. A fuzzy closure operator \(\psi\) on \(X\) is said to be fuzzy topological, if \(\psi(\psi(A)) = \psi(A)\) for all \(A \in L^X\).

Proposition 2.6. If \((X, d)\) is a fuzzy pseudo metric space, then

(a) The function \(\{x_l \rightarrow d(x_l, A)\} : (X, d) \rightarrow R\) is continuous for each nonzero fuzzy set \(A\) in \(X\).

(b) \((X, d)\) is a fuzzy topological space. That is, the fuzzy closure operator \(\psi\) induced by \(d\) is fuzzy topological.

(c) Every open (closed) sphere in \((X, d)\) is an open (closed) subset of \((X, d)\).

Proof.

(a) It follows from (i) of Preposition 2.3 that each function of (a) is a Lipschitz continuous function. Therefore each mapping of (a) is continuous by Preposition 2.1.

(b) To prove (b), it is sufficient to show that \(\psi(A)\) is closed for each nonzero fuzzy subset \(A\) of \(L^X\) where \(\psi\) is the fuzzy closure operator induced by \(d\). If \(f\) is the function of (a) corresponding to \(A\), then clearly \(\psi(A) = f^{-1}(\{0\})\). Since \(f\) is continuous and \(\{0\}\) is a closed subset of \(R\), \(\psi(A)\) is a closed subset of \((X, d)\), by Definition 1.12.

(c) The open (closed) \(r\)-sphere about a fuzzy point \(x_l\) is clearly the inverse image of the open interval \((-r, r)\) (the closed interval \([-r, r]\)) of \(R\) under the function \(\{y_m \rightarrow d(y_m, x_l)\} : (X, d) \rightarrow R\) which is continuous by (a) because \(d(y_m, x_l) = d(y_m, \{x_l\})\). Now the statement follows by Definition 1.12.
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References


