Intuitionistic Fuzzy Module over intuitionistic Fuzzy Ring

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Abstract

In this paper the concept of an intuitionistic fuzzy module over an intuitionistic fuzzy ring is introduced and discussed.

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Introduction

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2] as a generalization of that of fuzzy sets and it is a very effective tool to study the case of vagueness. Further many researches applied this notion in various branches of mathematics especially in algebra and defined intuitionistic fuzzy subgroups, intuitionistic fuzzy subrings, and intuitionistic fuzzy sublattices, intuitionistic fuzzy submodules and so forth. In this paper the concept of intuitionistic fuzzy module over an intuitionistic fuzzy ring is defined and discussed.

Preliminaries

In this section we recall some definitions and results which will be used later. In this paper, a ring $R$ will be a commutative ring with unity and $R$-modules $M$ will be a unitary module unless otherwise stated.

Definition 2.1 [1] Let $X$ be a fixed non-empty set. An Intuitionistic fuzzy set (IFS) $A$ of $X$ is an object of the following form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and degree of non-
membership of the element \( x \in X \) respectively and for any \( x \in X \), we have \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \).

**Remark 2.2**
(i): When \( \mu_A(x) + \nu_A(x) = 1 \), i.e. when \( \nu_A(x) = 1 - \mu_A(x) = \mu^c_A(x) \). Then \( A \) is called fuzzy set.
(ii) We write \( A = (\mu_A, \nu_A) \) to denote the IFS \( A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \).

**Definition 2.3** [5]: Let \( R \) be a ring. An IFS \( A = (\mu_A, \nu_A) \) of \( R \) is said to be intuitionistic fuzzy subring of \( R \) (In short IFSR) of \( R \) if

\[
\begin{align*}
(i) & \quad \mu_A(x - y) \geq \mu_A(x) \land \mu_A(y) \\ (ii) & \quad \mu_A(xy) \geq \mu_A(x) \land \mu_A(y) \\ (iii) & \quad \nu_A(x - y) \leq \nu_A(x) \lor \nu_A(y) \\ (iv) & \quad \nu_A(xy) \leq \nu_A(x) \lor \nu_A(y), \quad \forall x, y \in R
\end{align*}
\]

**Definition 2.4** [5]: An IFS \( A = (\mu_A, \nu_A) \) of a ring \( R \) said to be intuitionistic fuzzy Ideal (IFI) of \( R \) if

\[
\begin{align*}
(i) & \quad \mu_A(x - y) \geq \mu_A(x) \land \mu_A(y) \\ (ii) & \quad \mu_A(xy) \geq \mu_A(x) \land \mu_A(y) \\ (iii) & \quad \nu_A(x - y) \leq \nu_A(x) \lor \nu_A(y) \\ (iv) & \quad \nu_A(xy) \leq \nu_A(x) \lor \nu_A(y), \quad \forall x, y \in R
\end{align*}
\]

**Proposition 2.5** [5]: Let \( A = (\mu_A, \nu_A) \) be IFSR of \( R \), then

\[
\begin{align*}
\mu_A(0) \geq & \mu_A(x) \quad \text{and} \quad \nu_A(0) \leq & \nu_A(x), \quad \forall x \in R
\end{align*}
\]

If \( R \) is a ring with unity, then \( \mu_A(1) \leq \mu_A(x) \) and \( \nu_A(1) \geq \nu_A(x) \), for all \( x \in R \)

**Definition 2.7** [8, 10]: Let \( M \) be a module over a ring \( R \). An IFS \( A = (\mu_A, \nu_A) \) of \( M \) is called intuitionistic fuzzy submodule (IFSM) if

\[
\begin{align*}
\mu_A(0) = 1, \nu_A(0) = 0 \\
\mu_A(x + y) \geq & \mu_A(x) \land \mu_A(y) \quad \text{and} \quad \nu_A(x + y) \leq & \nu_A(x) \lor \nu_A(y), \quad \forall x, y \in M \\
\mu_A(rx) \geq & \mu_A(x) \quad \text{and} \quad \nu_A(rx) \leq & \nu_A(x), \quad \forall x \in M, r \in R
\end{align*}
\]

**Proposition 2.8** [10]: Let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) are two IFSM’s of \( M \), then \( A \cap B \) is also IFSM of \( M \).

**Definition 2.9** [9]: Let \( X \) and \( Y \) be two non-empty sets and \( f: X \to Y \) be a mapping. Let \( A \) and \( B \) be IFS’s of \( X \) and \( Y \) respectively. Then the image of \( A \) under the map \( f \) is denoted by \( f(A) \) and is defined as

\[
\begin{align*}
f(A)(y) = \begin{pmatrix} \mu_{f(x)}(y) \quad \nu_{f(x)}(y) \end{pmatrix}, \quad \text{where}
\end{align*}
\]
\[ \mu_{f(A)}(y) = \begin{cases} \bigvee \{ \mu_A(x) : x \in f^{-1}(y) \} & \text{and} \quad \nu_{f(A)}(y) = \bigwedge \{ \nu_A(x) : x \in f^{-1}(y) \} \\ 0 & \text{otherwise} \end{cases} \]

Also the pre-image of \( B \) under \( f \) is denoted by \( f^{-1}(B) \) and is defined as
\[ f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)) \]
where \( \mu_{f^{-1}(B)}(x) = \mu_B(f(x)) \) and \( \nu_{f^{-1}(B)}(x) = \nu_B(f(x)) \)
i.e. \( f^{-1}(B)(x) = (\mu_B(f(x)), \nu_B(f(x))) \)

**Remark 2.10**
Clearly \( \mu_{f(A)}(f(x)) \geq \mu_A(x) \) and \( \nu_{f(A)}(f(x)) \leq \nu_A(x) \), for any \( x \in X \).

**Intuitionistic Fuzzy module over intuitionistic Fuzzy Ring**
We now introduce the concept of an intuitionistic fuzzy module over an intuitionistic fuzzy ring.

**Definition 3.1** Let \( (M, R, A_R, A_M) \) be quaternary group, where \( M \) is an \( R \)-module, \( A_R \) is an intuitionistic fuzzy subring of \( R \), \( A_M \) is intuitionistic fuzzy submodule of \( M \), if for all \( x, y \in M, r \in R \), we have
\begin{align*}
(i) \quad & \mu_{A_R}(x+y) \geq \mu_{A_R}(x) \wedge \mu_{A_R}(y) \quad \text{and} \quad \nu_{A_R}(x+y) \leq \nu_{A_R}(x) \vee \nu_{A_R}(y) \\
(ii) \quad & \mu_{A_M}(-x) = \mu_{A_R}(x) \quad \text{and} \quad \nu_{A_M}(-x) = \nu_{A_R}(x) \\
(iii) \quad & \mu_{A_R}(0) = 1 \quad \text{and} \quad \nu_{A_R}(0) = 0 \\
(iv) \quad & \mu_{A_M}(rx) \geq \mu_{A_R}(r) \vee \mu_{A_R}(x) \quad \text{and} \quad \nu_{A_M}(rx) \leq \nu_{A_R}(r) \wedge \nu_{A_R}(x) 
\end{align*}

Then \( (M, R, A_R, A_M) \) is called a intuitionistic fuzzy submodule of \( M \) over an intuitionistic fuzzy ring \( R \). In brief \( A_M \) is called \( A_R \)-IFIF module or simply aIFIF-module.

**Example 3.2:** Let \( R = (Z_2, +_2, \times_2) \), where \( Z_2 = \{0, 1\} \).
Let \( A_R = (\mu_{A_R}, \nu_{A_R}) \), where
\[ \mu_{A_R}(0) = 0.7, \quad \mu_{A_R}(1) = 0.5, \quad \nu_{A_R}(0) = 0.2, \quad \nu_{A_R}(1) = 0.6 \]
Clearly, \( A_R \) is IFSR of \( R \). Let \( M = Z_2 \) and \( A_M = (\mu_{A_M}, \nu_{A_M}) \), where
\[ \mu_{A_M}(0) = 1, \quad \mu_{A_M}(1) = 0.6, \quad \nu_{A_M}(0) = 0, \quad \nu_{A_M}(1) = 0.5 \]
Clearly, \( A_M \) is IFSM of \( M \).

Moreover, it can be easily checked that \( A_M \) is \( A_R \)-IFIF submodule of \( M \).
Now, if we take \( M = Z_2 \) and \( A_M = (\mu_{A_M}, \nu_{A_M}) \), where
Clearly, \( \text{A}_M \) is IFSM of \( M \).

But \( \text{A}_M \) is not an \( AR \)-IFIF submodule of \( M \) as
\[
\mu_{A_R}(1.1) = \mu_{A_R}(1) = 0 < \mu_{A_R}(1) \lor \mu_{A_R}(1) = 0.5 \lor 0 = 0.5.
\]

**Theorem 3.3** Let \( A_R \) be an intuitionistic fuzzy subring of a ring \( R \), if there exist \( a \in R \) such that \( \mu_{A_R}(a) = 1 \) and \( \nu_{A_R}(a) = 0 \), then \( A_R \) is an \( AR \)-IFIF module if and only if \( A_R \) is intuitionistic fuzzy ideal of \( R \).

**Proof.** The Necessary part is trivial, for sufficient part, we have
\[
\mu_{A_R}(0) = \mu_{A_R}(a-a) \geq \mu_{A_R}(a) \land \mu_{A_R}(a) = \mu_{A_R}(a) = 1 \quad \text{and}
\nu_{A_R}(0) = \nu_{A_R}(a-a) \leq \nu_{A_R}(a) \lor \nu_{A_R}(a) = \nu_{A_R}(a) = 0.
\]

**Proposition 3.4:** Let \( A_M \) be an \( AR \)-IFIF submodule of \( M \). If \( r \in R \) (with unity) is invertible, then \( \mu_{A_R}(rx) = \mu_{A_R}(x) \) and \( \nu_{A_R}(rx) = \nu_{A_R}(x) \), for all \( x \in M \)

**Proof.** Since \( r \in R \) is invertible, then we have for all \( x \in M \)
\[
\mu_{A_R}(x) = \mu_{A_R}(r^{-1}rx) \geq \mu_{A_R}(rx) \geq \mu_{A_R}(x). \quad \text{Similarly, } \nu_{A_R}(x) = \nu_{A_R}(r^{-1}rx) \leq \nu_{A_R}(rx) \leq \nu_{A_R}(x)
\]
Thus \( \mu_{A_R}(rx) = \mu_{A_R}(x) \) and \( \nu_{A_R}(rx) = \nu_{A_R}(x) \).

**Proposition 3.5** Let \( A_M \) be an \( AR \)-IFIF submodule of \( M \), then

(i) \( \mu_{A_R}(0) \geq \mu_{A_R}(x) \) and \( \nu_{A_R}(0) \leq \nu_{A_R}(x) \) for all \( x \in M \)

(ii) If \( M \) is unitary \( R \)-module, then \( \mu_{A_R}(x) \geq \mu_{A_R}(1) \) and \( \nu_{A_R}(x) \leq \nu_{A_R}(1) \), for all \( x \in M \)

**Proof.** (i) Since \( \mu_{A_R}(0) = \mu_{A_R}(x-x) \geq \mu_{A_R}(x) \land \mu_{A_R}(x) = \mu_{A_R}(x) \) and
\[
\nu_{A_R}(0) = \nu_{A_R}(x-x) \leq \nu_{A_R}(x) \lor \nu_{A_R}(x) = \nu_{A_R}(x).
\]
(ii) Since \( M \) is unitary \( R \)-module, therefore \( 1.x = x \), for all \( x \in M \). So we have
\[
\mu_{A_R}(x) = \mu_{A_R}(1.x) \geq \mu_{A_R}(1) \lor \mu_{A_R}(x) \Rightarrow \mu_{A_R}(x) \geq \mu_{A_R}(1).
\]
Similarly, \( \nu_{A_R}(x) = \nu_{A_R}(1.x) \leq \nu_{A_R}(1) \lor \nu_{A_R}(x) \Rightarrow \nu_{A_R}(x) \leq \nu_{A_R}(1) \)
Thus \( \mu_{A_R}(x) \geq \mu_{A_R}(1) \) and \( \nu_{A_R}(x) \leq \nu_{A_R}(1) \).

**Theorem 3.6** Let \( A_M \) be an intuitionistic fuzzy submodule of an \( R \)-module \( M \) and \( A_R \) be an intuitionistic fuzzy subring of \( R \). Then \( A_M \) is an \( AR \)-IFIF submodule of \( M \) if and only if \( A_M \) satisfies the following conditions:

(i) \( \mu_{A_R}(0) = 1 \) and \( \nu_{A_R}(0) = 0 \)

(ii) \( \mu_{A_R}(rx + sy) \geq \{ \mu_{A_R}(r) \lor \mu_{A_R}(x) \} \land \{ \mu_{A_R}(s) \lor \mu_{A_R}(y) \} \) and
\[
\nu_{A_R}(rx + sy) \leq \{ \nu_{A_R}(r) \land \nu_{A_R}(x) \} \lor \{ \nu_{A_R}(s) \land \nu_{A_R}(y) \}, \quad \text{for all } x, y \in M \text{ and } r, s \in R \)
**Proof.** Firstly, let \( A_M \) be an \( A_R - \text{IFIF submodule} \) of \( M \). Then condition (i) hold, also \( \mu_{A_\alpha}(rx + sy) \geq \mu_{A_\beta}(rx) \wedge \mu_{A_\gamma}(sy) \geq \mu_{A_\delta}(r) \vee \mu_{A_\epsilon}(y) \) and

\[
\nu_{A_\phi}(rx + sy) \leq \nu_{A_\chi}(rx) \vee \nu_{A_\psi}(sy) \leq \nu_{A_\iota}(r) \wedge \nu_{A_\kappa}(y)
\]

Conversely, let the above conditions hold, put \( r = s = 1 \) in (ii), we get

\[
\mu_{A_\alpha}(1x + 1y) \geq \mu_{A_\beta}(1) \vee \mu_{A_\gamma}(1) \wedge \mu_{A_\delta}(0) = \mu_{A_\epsilon}(x) \wedge \mu_{A_\zeta}(y)
\]

and

\[
\nu_{A_\phi}(1x + 1y) \leq \nu_{A_\chi}(1) \wedge \nu_{A_\psi}(1) = \nu_{A_\iota}(x) \vee \nu_{A_\kappa}(y)
\]

Now \( \mu_{A_\alpha}(rx) = \mu_{A_\beta}(rx + 1.0) \geq \mu_{A_\gamma}(r) \wedge \mu_{A_\delta}(x) \wedge \mu_{A_\epsilon}(0) = \mu_{A_\zeta}(r) \wedge \mu_{A_\iota}(x) \wedge \mu_{A_\kappa}(0) \)

and

\[
\nu_{A_\phi}(rx) = \nu_{A_\chi}(rx + 1.0) \leq \nu_{A_\psi}(r) \wedge \nu_{A_\iota}(x) \leq \nu_{A_\kappa}(r) \wedge \nu_{A_\theta}(x) \wedge \nu_{A_\xi}(0) = \nu_{A_\iota}(r) \wedge \nu_{A_\kappa}(x) \leq \nu_{A_\kappa}(r) \wedge \nu_{A_\iota}(x) \wedge \nu_{A_\theta}(0) = \nu_{A_\kappa}(r) \wedge \nu_{A_\iota}(x) = \nu_{A_\kappa}(r) \wedge \nu_{A_\iota}(x)
\]

Similarly, we can show that \( \mu_{A_\alpha}(\neg x) = \mu_{A_\beta}(\neg x) \) and \( \nu_{A_\phi}(\neg x) = \nu_{A_\iota}(\neg x) \).

**Hence** \( A_M \) is an \( A_R - \text{IFIF submodule} \) of \( M \).

**Proposition 3.7** If \( A_M \) and \( B_M \) be two \( (A \cap B)_R - \text{IFIF submodules} \) of \( M \), then \( (A \cap B)_M \) is \( (A \cap B)_R - \text{IFIF submodule} \) of \( M \).

**Proof**

Now \( \mu_{A_B}(0) = \mu_{A_A}(0) \wedge \mu_{A_B}(0) = 1 \wedge 1 = 1 \) and \( \nu_{A_B}(0) = \nu_{A_A}(0) \vee \nu_{A_B}(0) = 0 \vee 0 = 0 \)

Let \( x, y \in M \) and \( r, s \in R \), we have

\[
\mu_{A_B}(rx + sy) = \mu_{A_A}(rx + sy) \wedge \mu_{A_B}(rx + sy)
\]

\[
\geq [\mu_{A_B}(r) \vee \mu_{A_A}(s) \wedge \mu_{A_B}(y)] \wedge [\mu_{A_B}(r) \wedge \mu_{A_A}(s) \wedge \mu_{A_B}(y)]
\]

\[
= [\mu_{A_B}(r) \vee \mu_{A_A}(s) \wedge \mu_{A_B}(y)] \wedge [\mu_{A_B}(r) \wedge \mu_{A_A}(s) \wedge \mu_{A_B}(y)]
\]

\[
= [\mu_{A_B}(r) \vee \mu_{A_A}(s) \wedge \mu_{A_B}(y)] \wedge [\mu_{A_B}(r) \wedge \mu_{A_A}(s) \wedge \mu_{A_B}(y)]
\]

Similarly, we can show that

\[
v_{(A \cap B)_B}(rx + sy) \leq v_{(A \cap B)_A}(rx) \vee v_{(A \cap B)_B}(sy) \leq v_{(A \cap B)_B}(sy) \wedge v_{(A \cap B)_A}(rx)
\]

**Hence** \( (A \cap B)_M \) is \( (A \cap B)_R - \text{IFIF submodule} \) of \( M \).

**Theorem 3.8** Let \( A_R \) be intuitionistic fuzzy subring of a ring \( R \), \( M \) be a \( R \)-module and \( A_M \) be intuitionistic fuzzy submodule of \( M \), then \( A_M \) is \( A_R - \text{IFIF submodule} \) of \( M \) if and only if

(i) if there exists \( x \in M \) such that \( \mu_{A_\alpha}(x) = 1 \) and \( \nu_{A_\phi}(x) = 0 \)

(ii) for all \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \), \( C_{A_\alpha}(A_\beta) \) is left \( R \)-module , where \( \mu_{A_\alpha}(0) \geq \alpha, \nu_{A_\phi}(0) \leq \beta \)

(iii) for all \( x \in C_{A_\alpha}(A_\beta) \) and \( r \in C_{A_\alpha}(A_\beta) \), there exists \( \gamma, \eta \in [0,1] \) with \( \gamma + \eta \leq 1 \)

and \( \gamma \geq \alpha \vee \delta, \eta \leq \beta \wedge \theta \) such that \( rx \in C_{\gamma \eta}(A_\alpha) \).
Proof: Firstly, let $A_M$ is $A_R$- IFIF submodule of $M$, then (i) is Obvious (ii) follows from ( [ 10 ], Theorem.(3.2)) and for (iii) let $x \in C_{\alpha,\beta}(A_M)$ and $r \in C_{\delta,\theta}(A_M)$

\[\Rightarrow \mu_{A_M}(x) \geq \alpha, \mu_{A_M}(x) \leq \beta \text{ and } \mu_{A_M}(r) \geq \delta, \nu_{A_M}(r) \leq \theta\]

Let \(\gamma = \mu_{A_M}(rx) \geq \mu_{A_M}(r) \vee \mu_{A_M}(x) \geq \alpha \vee \delta\) and \(\eta = \nu_{A_M}(rx) \leq \nu_{A_M}(r) \wedge \nu_{A_M}(x) \leq \beta \wedge \theta\)

i.e. there exists $x, y \in [0,1]$ with $\gamma + \eta \leq 1$ and $\gamma \geq \alpha \vee \delta$, $\eta \leq \beta \wedge \theta$ such that $rx \in C_{\gamma,\eta}(A_M)$

Conversely, let $x, y \in M$, $r \in R$, let $\mu_{A_M}(x) \wedge \mu_{A_M}(y) = \alpha$ and $\nu_{A_M}(x) \vee \nu_{A_M}(y) = \beta$.

Then there exists $x, y \in M$, and $r, s \in R$ be any elements. Then we have

\[\mu_{A_M}(rx + sy) = \mu_{A_M}(f(rx + sy)) = \mu_{A_M}(rf(x) + sf(y))\]

\[\geq \left\{ \mu_{A_M}(r) \lor \mu_{A_M}(f(x)) \right\} \wedge \left\{ \mu_{A_M}(s) \lor \mu_{A_M}(f(y)) \right\}\]

\[= \left\{ \mu_{A_M}(r) \lor \mu_{f^{-1}(A_M)}(x) \right\} \wedge \left\{ \mu_{A_M}(s) \lor \mu_{f^{-1}(A_M)}(y) \right\}\]

also, \(\nu_{f^{-1}(A_M)}(rx + sy) = \nu_{A_M}(f(rx + sy)) = \nu_{A_M}(rf(x) + sf(y))\)

\[\leq \left\{ \nu_{A_M}(r) \lor \nu_{A_M}(f(x)) \right\} \wedge \left\{ \nu_{A_M}(s) \lor \nu_{A_M}(f(y)) \right\}\]

\[= \left\{ \nu_{A_M}(r) \lor \nu_{f^{-1}(A_M)}(x) \right\} \wedge \left\{ \nu_{A_M}(s) \lor \nu_{f^{-1}(A_M)}(y) \right\}\]

Further, $\mu_{f^{-1}(A_M)}(0) = \mu_{A_M}(f(0)) = \mu_{A_N}(0') = 1$. Similarly, $\nu_{f^{-1}(A_M)}(0) = \nu_{A_M}(f(0)) = \nu_{A_N}(0') = 0$.

Hence $f^{-1}(A_N)$ is an $A_R$- IFIF submodule of $M$.

Proposition 3.9 Let $f: M \rightarrow N$ be an homomorphism from an $R$-module $M$ into an $R$-module $N$. Let $A_N$ be an $A_R$ –IFIF sub module of $N$. Then $f^{-1}(A_N)$ is an $A_R$- IFIF submodule of $M$.

Proof. Let $x, y \in M$ and $r, s \in R$ be any elements. Then we have

\[\mu_{f^{-1}(A_N)}(rx + sy) = \mu_{A_N}(f(rx + sy)) = \mu_{A_N}(rf(x) + sf(y))\]

\[\geq \left\{ \mu_{A_N}(r) \lor \mu_{A_N}(f(x)) \right\} \wedge \left\{ \mu_{A_N}(s) \lor \mu_{A_N}(f(y)) \right\}\]

\[= \left\{ \mu_{A_N}(r) \lor \mu_{f^{-1}(A_N)}(x) \right\} \wedge \left\{ \mu_{A_N}(s) \lor \mu_{f^{-1}(A_N)}(y) \right\}\]

also, \(\nu_{f^{-1}(A_N)}(rx + sy) = \nu_{A_N}(f(rx + sy)) = \nu_{A_N}(rf(x) + sf(y))\)

\[\leq \left\{ \nu_{A_N}(r) \lor \nu_{A_N}(f(x)) \right\} \wedge \left\{ \nu_{A_N}(s) \lor \nu_{A_N}(f(y)) \right\}\]

\[= \left\{ \nu_{A_N}(r) \lor \nu_{f^{-1}(A_N)}(x) \right\} \wedge \left\{ \nu_{A_N}(s) \lor \nu_{f^{-1}(A_N)}(y) \right\}\]

Hence $f^{-1}(A_N)$ is an $A_R$- IFIF submodule of $M$.

Proposition 3.10 Let $f: M \rightarrow N$ be an homomorphism from an $R$-module $M$ onto an $R$-module $N$. Let $A_M$ be an $A_R$ –IFIF submodule of $M$. Then $f(A_M)$ is an $A_R$- IFIF submodule of $N$.

Proof. Let $x, y \in N$ and $r, s \in R$ be any elements. As $f$ is onto, therefore there exists $x, y \in M$ such that $f(x) = x'$ and $f(y) = y'$.
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Now \( \mu_{f(A)}(rx' + sy') = \mu_{f(A)}(rf(x) + sf(y)) = \mu_{f(A)}(f(rx + sy)) \)

\begin{align*}
&= \sup \{ \mu_{\alpha}(z) : z \in f^{-1}(f(rx + sy)) \} \\
&\geq \mu_{\alpha}(rx + sy) \\
&\geq \{ \mu_{\alpha}(r) \lor \mu_{\alpha}(x) \} \land \{ \mu_{\alpha}(s) \lor \mu_{\alpha}(y) \} \\
&\geq \{ \mu_{\alpha}(r) \lor \mu_{f(A)}(f(x)) \} \land \{ \mu_{\alpha}(s) \lor \mu_{f(A)}(f(y)) \} \quad \text{[using Remark (2.10)]} \\
&= \{ \mu_{\alpha}(r) \lor \mu_{f(A)}(x') \} \land \{ \mu_{\alpha}(s) \lor \mu_{f(A)}(y') \}
\end{align*}

Similarly, \( \nu_{f(A)}(rx' + sy') = \nu_{f(A)}(rf(x) + sf(y)) = \nu_{f(A)}(f(rx + sy)) \)

\begin{align*}
&= \inf \{ \nu_{\alpha}(z) : z \in f^{-1}(f(rx + sy)) \} \\
&\leq \nu_{\alpha}(rx + sy) \\
&\leq \{ \nu_{\beta}(r) \land \nu_{\alpha}(x) \} \lor \{ \nu_{\beta}(s) \land \nu_{\alpha}(y) \} \\
&\leq \{ \nu_{\beta}(r) \land \nu_{f(A)}(f(x)) \} \lor \{ \nu_{\beta}(s) \land \nu_{f(A)}(f(y)) \} \quad \text{[using Note (2.7)]} \\
&= \{ \nu_{\beta}(r) \land \nu_{f(A)}(x') \} \lor \{ \nu_{\beta}(s) \land \nu_{f(A)}(y') \}
\end{align*}

Further, \( \mu_{f(A)}(0') = \mu_{f(A)}(f(0)) \geq \mu_{\alpha}(0) = 1 \), but \( \mu_{f(A)}(0') \leq 1 \) (always). So \( \mu_{f(A)}(0') = 1 \).

Similarly, \( \nu_{f(A)}(0') = \nu_{f(A)}(f(0)) \leq \nu_{\alpha}(0) = 0 \), but \( \nu_{f(A)}(0') \geq 0 \) (always). So \( \nu_{f(A)}(0') = 0 \).

Hence \( f(A) \) is an \( \text{IFIF} \) submodule of \( N \).

References


