Generalized Regular Interval Valued Fuzzy Matrices

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Abstract

In this paper, the concept of k – regular Interval Valued Fuzzy Matrix (IVFM) as a generalization of regular Interval Valued Fuzzy Matrix and as an extension of k – regular fuzzy matrix is introduced and some basic properties of a k – regular IVFM are derived.

Keywords: Fuzzy matrix, k – regular matrix, Interval Valued Fuzzy Matrix, Generalized Inverse.

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Introduction

We deal with Interval Valued Fuzzy Matrices (IVFM) that is, matrices whose entries are intervals and all the intervals are subintervals of the interval [0,1]. Thomason introduced fuzzy matrices and discussed about the convergence of powers of a fuzzy matrix [5]. Recently the concept of IVFM a generalization of fuzzy matrix was introduced and developed by Shyamal and Pal [4], by extending the max.min operations on fuzzy algebra $F = [0,1]$, for elements $a,b \in F$, $a+b = \max\{a,b\}$ and $a.b = \min\{a,b\}$. In [2], Meenakshi and Kaliraja have represented an IVFM as an interval matrix of its lower and upper limit fuzzy matrices. In [3], Meenakshi and Jenita have introduced the concept of k – regular fuzzy matrix as a generalization of a regular fuzzy matrix and discussed about various inverses associated with a k – regular fuzzy matrix. A matrix $A \in F_n$, the set of all nxn fuzzy matrices is said to be right(left) k-regular if there exists $X (Y) \in F_n$, such that $A^k X A = A^k$ ($A^k (A^k Y) = A^k$), $X(Y)$ is called a right (left) k-g inverse of $A$, where $k$ is a positive integer. In [3], it is exhibited that right and left k – g inverses are distinct. In particular for $k = 1$, it reduces to a regular matrix and set of all its $g$ – inverses. By a k- regular matrix, we...
mean that it is either right or left k-regular. If A is k-regular, then it is h-regular for all $h \geq k$.

In this paper, we introduce the concept of k–regular Interval Valued Fuzzy Matrix as a generalization of regular Interval Valued Fuzzy Matrix and as an extension of $k$–regular fuzzy matrix. In section 2, we present the basic definitions, notations on IVFM and required results on $k$-regular fuzzy matrices and regular IVFM. In section 3, we introduce the concept of $k$–regular IVFM. The row and column ranks of a $k$–regular IVFM are determined.

**Preliminaries**

In this section, some basic definitions and results needed are given. Let $(IVFM)_n$ denotes the set of all $n \times n$ Interval Valued Fuzzy Matrices.

**Definition 2.1**

An Interval Valued Fuzzy Matrix (IVFM) of order $m \times n$ is defined as $A = (a_{ij})_{m \times n}$, where $a_{ij} = [a_{ijL}, a_{ijU}]$, the $ij$th element of $A$ is an interval representing the membership value. All the elements of an IVFM are intervals and all the intervals are the subintervals of the interval $[0,1]$.

For $A = (a_{ij}) = ([a_{ijL}, a_{ijU}])$ and $B = (b_{ij}) = ([b_{ijL}, b_{ijU}])$ of order $m \times n$ their sum denoted as $A + B$ defined as,

$$A + B = (a_{ij} + b_{ij}) = ([a_{ijL} + b_{ijL}, a_{ijU} + b_{ijU}])$$ (2.1)

For $A = (a_{ij}) = ([a_{ijL}, a_{ijU}])$ and $B = (b_{ij}) = ([b_{ijL}, b_{ijU}])$ of order $m \times n$ their product denoted as $AB$ is defined as,

$$AB = (c_{ij}) = \left\{ \begin{array}{l}
\sum_{k=1}^{n} a_{ik} b_{kj} \\
\end{array} \right\} 
\begin{array}{l}
i = 1, 2, \ldots, m \\
j = 1, 2, \ldots, p 
\end{array}$$

$$= \left[ \left\{ \sum_{k=1}^{n} a_{ikL} \cdot b_{kjL} \right\}, \left\{ \sum_{k=1}^{n} a_{ikU} \cdot b_{kjU} \right\} \right]
\begin{array}{l}
i = 1, 2, \ldots, m \\
j = 1, 2, \ldots, p 
\end{array}$$ (2.2)

In particular if $a_{ijL} = a_{ijU}$ and $b_{ijL} = b_{ijU}$ then (2.2) reduces to the standard max. min composition of Fuzzy Matrices [1].

$A \leq B$ if and only if $a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$

In [2], representation of an IVFM is introduced as in the following:

**Definition 2.2**

For a pair of Fuzzy Matrices $E = (e_{ij})$ and $F = (f_{ij})$ in $F_{m,n}$ such that $E \leq F$, let us define the interval matrix denoted as $[E, F]$, whose $ij$th entry is the interval with lower limit $e_{ij}$ and upper limit $f_{ij}$, that is $([e_{ijL}, f_{ijU})]$. In particular for $E = F$, IVFM $[E,E]$ reduces to $E \in F_{m,n}$.

For $A = (a_{ij}) = ([a_{ijL}, a_{ijU}]) \in (IVFM)_{m \times n}$, let us define $A_L = (a_{ijL})$ and $A_U = (a_{ijU})$.

Clearly $A_L$ and $A_U$ belongs to $F_{m,n}$ such that $A_L \leq A_U$ and from Definition (2.2) $A$ can be written as $A = [A_L, A_U]$ ...(2.3).
For $A \in (IVFM)_{mn}$, $A^T$, $A_{ir}$, $A_{sj}$, $R(A)$, $C(A)$, $\rho_r(A)$, $\rho_c(A)$ denotes the transpose of $A$, $i^{th}$ row of $A$, $j^{th}$ column of $A$, row space of $A$, column space of $A$, row rank of $A$, column rank of $A$ respectively.

In the sequel we shall make use of the following results on Interval Valued Fuzzy Matrices found in [2].

**Lemma 2.3**
For $A = [A_L, A_U] \in (IVFM)_{mn}$ and $B = [B_L, B_U] \in (IVFM)_{np}$, the following hold.
(i) $A^T = [A_L^T, A_U^T]$
(ii) $AB = [A_LB_L, A_UB_U]$

**Lemma 2.4**
For $A, B \in (IVFM)_{mn}$
(i) $R(B) \subseteq R(A) \iff B =XA$ for some $X \in (IVFM)_m$
(ii) $C(B) \subseteq C(A) \iff B =AY$ for some $Y \in (IVFM)_n$

**Lemma 2.5**
For $A \in (IVFM)_{mn}$ and $B \in (IVFM)_{np}$, the following hold.
(i) $R(AB) \subseteq R(A)$
(ii) $C(AB) \subseteq C(B)$

**Lemma 2.6**
Let $A = [A_L, A_U]$ be an $(IVFM)_{mn}$
Then, (i) $R(A) = [R(A_L), R(A_U)] \in (IVFM)_{1n}$
(ii) $C(A) = [C(A_L), C(A_U)] \in (IVFM)_{1m}$

**Generalized Regular Interval Valued Fuzzy Matrices**
In this section, we introduce the concept of $k$ – regular IVFM. The row and column ranks of a $k$ – regular IVFM are determined as a generalization of the results found in [2] and [3].

**Definition 3.1**
A matrix $A \in (IVFM)_n$ is said to be right $k$ – regular if there exist a matrix $X \in (IVFM)_n$, such that $A^k X A = A^k$, for some positive integer $k$. $X$ is called a right $k$ – g inverse of $A$. Let $A_{r\{k\}} = \{X / A^k X A = A^k \}$.

**Definition 3.2**
A matrix $A \in (IVFM)_n$ is said to be left $k$ – regular if there exist a matrix $Y \in (IVFM)_n$, such that $A Y A^k = A^k$, for some positive integer $k$. $Y$ is called a left $k$ – g inverse of $A$. Let $A_{l\{k\}} = \{Y / A Y A^k = A^k \}$.
In general, right k-regular IVFM is different from left k-regular IVFM. Hence a right k-g inverse need not be a left k-g inverse. This is illustrated in the following example.

**Example 3.3**

Let us consider

\[
A = \begin{bmatrix}
0, & 0 \\
0, & 0 \\
0.5, & 1 \\
0.3, & 0.5
\end{bmatrix} \in (IVFM)_{3x3}.
\]

For this A, \(A^2 = \begin{bmatrix}
0.1, & 0.5 \\
0, & 0 \\
0.1, & 0.5 \\
0, & 0
\end{bmatrix}\)

\(A^3 = \begin{bmatrix}
0.1, & 0.5 \\
0, & 0 \\
0.1, & 0.5 \\
0, & 0
\end{bmatrix}\)

For \(X = \begin{bmatrix}
0.2, & 0.5 \\
0, & 0 \\
0, & 0 \\
0.4, & 0.5 \\
0, & 0 \\
0, & 0
\end{bmatrix}\)

\(A^3 \times A = A^3\). Hence A is 3-regular. For \(k = 3\), \(A^3 \times A = A^3\) but \(A \times A^3 \neq A^3\)

Hence X is a right 3-g inverse but not a left 3-g inverse.

**Remark 3.4**

In particular for \(k = 1\), Definitions (3.1) and (3.2) reduce to regular IVFM found in [2], and in the case \(A_L = A_U\), Definitions (3.1) and (3.2) reduce to right k-regular and left k-regular fuzzy matrix found in [3].

**Theorem 3.5**

Let \(A = [A_L, A_U] \in (IVFM)_n\). Then A is right k-regular IVFM \(\iff A_L \text{ and } A_U \in F_n\) are right k-regular.

**Proof**

Let \(A = [A_L, A_U] \in (IVFM)_n\).

Since A is right k-regular IVFM, there exists \(X \in (IVFM)_n\), such that \(A^k \times A = A^k\)

Let \(X = [X_L, X_U] \) with \(X_L, X_U \in F_n\)

Then by Lemma (2.3) (ii),

\(A^k \times A = A^k \Rightarrow [A_L, A_U]^k [X_L, X_U] [A_L, A_U] = [A_L, A_U]^k [A_L^k, A_U^k] [X_L, X_U] [A_L, A_U] = [A_L^k, A_U^k]

\([A_L^k, A_U^k] [X_L, X_U] [A_L^k, A_U^k] = [A_L^k, A_U^k]

\(A_L^k X_L A_L = A_L^k \text{ and } A_U^k X_U A_U = A_U^k\)
Therefore $A_L$ is right k – regular and $A_U$ is right k – regular $\in \mathbb{F}_n$. Thus A is right k – Regular IVFM $\Rightarrow A_L$ and $A_U \in \mathbb{F}_n$ are right k – regular.

Conversely, Suppose $A_L$ and $A_U \in \mathbb{F}_n$ are right k – regular, then $A_L k X_L A_L = A_L k$ and $A_U k X_U A_U = A_U k$ for some $X_L$ and $X_U \in \mathbb{F}_n$. $X_L \in (A_L)_{r\{1^k\}}$, $X_U \in (A_U)_{r\{1^k\}}$.

Since $A_L \leq A_U$, it is possible to choose at least one $V \in (A_L)_{r\{1^k\}}$ and $W \in (A_U)_{r\{1^k\}}$ such that $V \leq W$.

Let us define the interval valued fuzzy matrix $Z = [V, W]$. Then by Lemma(2.3)(ii),

\[
A^k Z A = [A_L^k, A_U^k] [V, W] [A_L, A_U] = [A_L^k V A_L, A_U^k W A_U] = [A_L^k, A_U^k] = A^k.
\]

Thus A is right k – regular IVFM. Hence the theorem.

**Theorem 3.6**

Let $A = [A_L, A_U] \in (IVFM)_n$. Then A is left k – regular IVFM $\iff A_L$ and $A_U \in \mathbb{F}_n$ are left k – regular.

**Proof**

This can be proved along the same lines as that of Theorem (3.5).

**Lemma 3.7**

For $A, B \in (IVFM)_n$, and a positive integer k, the following hold.

(i) If A is right k – regular and $R(B) \subseteq R(A^k)$ then, $B = BXA$ for each right k – g inverse X of A.

(ii) If A is left k – regular and $C(B) \subseteq C(A^k)$ then, $B = AYB$ for each left k – g inverse Y of A.

**Proof**

(i) Since $R(B) \subseteq R(A^k)$, by Lemma (2.4), there exists Z such that $B = ZA^k$. Since A is right k – regular, by Definition (3.1), $A^k X A = A^k$ for some $X \in A_{r\{1^k\}}$

Hence $B = ZA^k = ZA^k X A = BXA$. Thus (i) holds.

(ii) Since $C(B) \subseteq C(A^k)$, by Lemma (2.4), there exists U such that $B = A^k U$. Since A is left k – regular, by Definition (3.2), $A^k Y A^k = A^k$ for some $Y \in A_{l\{1^k\}}$

Hence $B = A^k U = AY A^k U = AYB$. Thus (ii) holds.

**Theorem 3.8**

For $A, B \in (IVFM)_n$, with $R(A) = R(B)$ and $R(A^k) = R(B^k)$ then A is right k – regular IVFM $\iff B$ is right k – regular IVFM.

**Proof**

Let A be a right k – regular IVFM satisfying $R(B^k) \subseteq R(A^k)$ and $R(A) \subseteq R(B)$. Since $R(B^k) \subseteq R(A^k)$, by Lemma (3.7), $B^k = B^k X A$ for each k – g inverse X of A. Since
R(A) ⊆ R(B), by Lemma (2.4), A = YB for some Y ∈ (IVFM)_n.

Substituting for A in $B^k = B^kXA$, we get,

$B^k = B^kXA = B^kXYB = B^kZB$ where XY = Z.

Hence B is a right k – regular IVFM.

Conversely, if B is a right k – regular IVFM satisfying $R(A^k) ⊆ R(B^k)$ and $R(B) ⊆ R(A)$, then A is right k – regular IVFM can be proved in the same manner. Hence the theorem.

**Theorem 3.9**

For A, B ∈ (IVFM)_n, with $C(A) = C(B)$ and $C(A^k) = C(B^k)$ then A is left k – regular IVFM $\iff$ B is left k – regular IVFM.

**Proof**

This is similar to Theorem (3.8) and hence omitted.

**Theorem 3.10**

For $A = [A_L, A_U]$ and $B = [B_L, B_U] ∈ (IVFM)_n$, with $R(A) = R(B)$ and $R(A^k) = R(B^k)$ then the following are equivalent:

- A is right k – regular IVFM
- $A_L$ and $A_U$ are right k – regular fuzzy matrices
- B is right k – regular IVFM
- $B_L$ and $B_U$ are right k – regular fuzzy matrices

**Proof**

(i) $\iff$ (ii) and (iii) $\iff$ (iv) are precisely Theorem (3.5).

(i) $\iff$ (iii) This follows from Theorem (3.8).

**Theorem 3.11**

For $A = [A_L, A_U]$ and $B = [B_L, B_U] ∈ (IVFM)_n$, with $C(A) = C(B)$ and $C(A^k) = C(B^k)$ then the following are equivalent:

- A is left k – regular IVFM
- $A_L$ and $A_U$ are left k – regular fuzzy matrices
- B is left k – regular IVFM
- $B_L$ and $B_U$ are left k – regular fuzzy matrices

**Proof**

(i) $\iff$ (ii) and (iii) $\iff$ (iv) are precisely Theorem (3.6).

(i) $\iff$ (iii) This follows from Theorem (3.9).

**Theorem 3.12**

Let $A = [A_L, A_U] ∈ (IVFM)_n$ and k be a positive integer, then the following hold.

If $X = [X_L, X_U] ∈ A_r{1^k}$ then $\rho_c(A_L^k) = \rho_c(A_L^k X_L)$, $\rho_c(A_U^k) = \rho_c(A_U^k X_U)$

and $\rho_r(A_L^k) ≤ \rho_r(X_L A_L) ≤ \rho_r(A_L)$, $\rho_r(A_U^k) ≤ \rho_r(X_U A_U) ≤ \rho_r(A_U)$
if $X \in A_{1}^{1k}$ then $\rho_c(A^k) = \rho_c(A^kX)$ and $\rho_r(A^k) \leq \rho_r(XA) \leq \rho_r(A)$
if $X = [X_L, X_U] \in A_{1}^{1k}$ then $\rho_c(A_L) = \rho_c([X_L A_L])$, $\rho_r(A_L) \leq \rho_r([X U A_U])$
and $\rho_c(A_U) \leq \rho_c([X_U A_U])$, $\rho_r(A_U) \leq \rho_r([X L A_L])$

Proof
Let $A = [A_L, A_U]$
since $X = [X_L, X_U] \in A_{1}^{1k}$, By Definition(3.1) and Lemma (2.3)(ii),
$A_{L}^k X_L A_L = A_{L}^k$ and $A_{U}^k X_U A_U = A_{U}^k$

By Lemma (2.5),
$C(A_L^k) = C(A_L^k X_L A_L) \subseteq C(A_L^k) \subseteq C(A_L^k) \quad (3.1)$
and $C(A_U^k) = C(A_U^k X_U A_U) \subseteq C(A_U^k) \subseteq C(A_U^k) \quad (3.2)$

Since $A_{L}^k X_L A_L = A_{L}^k$ and $A_{U}^k X_U A_U = A_{U}^k$ we have,
$A_{L}^k = A_{L}^k X_L A_L = A_{L}^k (X_L A_L)^k = \cdots = A_{L}^k (X_L A_L)^k$
$A_{U}^k = A_{U}^k X_U A_U = A_{U}^k (X_U A_U)^k = \cdots = A_{U}^k (X_U A_U)^k$
Therefore, $A_{L}^k = A_{L}^k (X_L A_L)^k$, Hence by Lemma (2.2),
$R(A_L^k) = R(A_L^k (X_L A_L)^k) \subseteq R((X_L A_L)^k) \subseteq R(X_L A_L) \subseteq R(A_L)$
Therefore $R(A_L^k) \subseteq R(X_L A_L) \subseteq R(A_L)$ \quad (3.3)

$\rho_c(A_L^k) \leq \rho_c(X_L A_L) \leq \rho_c(A_L)$
Similarly, $A_{U}^k = A_{U}^k (X_U A_U)^k$, Hence by Lemma (2.2),
$R(A_U^k) = R(A_U^k (X_U A_U)^k) \subseteq R((X_U A_U)^k) \subseteq R(X_U A_U) \subseteq R(A_U)$
Therefore $R(A_U^k) \subseteq R(X_U A_U) \subseteq R(A_U)$ \quad (3.4)

$\rho_r(A_L^k) \leq \rho_r(X_L A_L) \leq \rho_r(A_L)$
Thus (i) holds.
since $A = [A_L, A_U] \in (IVFM)_{n}$. From (3.1) and (3.2)
$C(A^k) = C([A_L^k, A_U^k])$
$= [C(A_L^k), C(A_U^k)] \quad [\text{By Lemma (2.6)}]$ 
$= [C(A_L^k X_L), C(A_U^k X_U)] = C(A^kX)$

And $\rho_c(A^k) = \rho_c(A^kX)$
Similarly From (3.3) and (3.4)
$R(A^k) = R([A_L^k, A_U^k])$
$= [R(A_L^k), R(A_U^k)] \quad [\text{By Lemma (2.6)}]$
$\subseteq [R(X_L A_L), R(X_U A_U)]$
$\subseteq [R(A_L), R(A_U)] = R(A)$

Therefore, $R(A^k) \subseteq R(XA) \subseteq R(A)$
$\rho_r(A^k) \leq \rho_r(XA) \leq \rho_r(A)$. Thus (ii) holds.
Proof is similar to that of (i) and hence omitted.
Proof is similar to that of (ii) and hence omitted.

References