

Common Fixed Point Theorems for Six Self Mappings in Fuzzy Metric Spaces under Compatibility of Type (α)

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Abstract

In this paper we prove a fixed point theorem for six self mappings by using compatibility of type (α) . Our results extend, generalized several fixed point theorems on metric and fuzzy metric spaces.

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Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh[26]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek[8] and George and Veeramani[4] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Recently, many authors have proved fixed point theorems involving fuzzy sets. Vasuki[24] investigated some fixed point theorems in fuzzy metric spaces for R-weakly commuting maps and Pant[16] introduced the notion of reciprocal continuity of mappings in metric spaces. Pant and Jha[18] proved an analogue of result given by Balasubramanian et al.[1]. S. Kutukcu et al [12] extended the result of Pant and Jha[18].

The purpose of this paper is to prove a common fixed point theorem in fuzzy metric spaces by compatible mappings of type (α) for six self maps which generalize the result of S. Kutukcu et al.[12].

Preliminaries

Definition 1.1: A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-

norm if $*$ is satisfying the following conditions:

- is commutative and associative,
- is continuous,
- $a*1 = a$ for all $a \in [0, 1]$,
- $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.2 ([4]): A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X, s, t > 0$.

$$(fm1) \quad M(x, y, t) > 0,$$

$$(fm2) \quad M(x, y, t) = 1 \text{ if and only if } x=y,$$

$$(fm3) \quad M(x, y, t) = M(y, x, t),$$

$$(fm4) \quad M(x, y, t) * M(y, z, s) \leq M(x, z, t+s),$$

$$(fm5) \quad M(x, y, \bullet): (0, \infty) \rightarrow [0, 1] \text{ is continuous,}$$

Then M is called a fuzzy metric on X . The function $M(x, y, t)$ denote the degree of nearness between x and y with respect to 't'. We identify $x=y$ with $M(x, y, t)=1$ for all $t>0$ and $M(x, y, t)=0$ with ∞ , and we can find some topological properties and examples of fuzzy metric spaces in paper of George and Veeramani[4].

Example 1.3 (Induced fuzzy metric [4]): Let (X, d) be a metric space. Define $a*b=ab$ for all $a, b \in [0, 1]$ and let M_d fuzzy sets on $X^2 \times (0, \infty)$ defined as follows, $M_d(x, y, t) = \frac{t}{t+d(x,y)}$ then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by the metric d , the standard fuzzy metric. On the other hand note that there exists no metric on X satisfying the above $M_d(x, y, t)$.

Definition 1.4 ([4]): Let $(X, M, *)$ be fuzzy metric space then

- a. A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \varepsilon < 1$ and $t>0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$; i.e $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$ for all $t>0$
- b. A sequence $\{x_n\}$ in X is Cauchy sequence if and only if for each $0 < \varepsilon < 1$ and $t>0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+p}, t) > 1 - \varepsilon$ for all $n \geq n_0$ and $p>0$;
i.e $M(x_n, x_{n+p}, t) \rightarrow 1$ as $n \rightarrow \infty$ for each $t>0$ and $p>0$.
- c. A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark 1.5: Since $*$ is continuous, it follows from (fm4) that the limit of the sequence in fuzzy metric space is uniquely determined.

Let $(X, M, *)$ is a fuzzy metric space with the following condition (fm6) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$.

Lemma 1.6([5]): For all $x, y \in X$, $M(x, y, \bullet)$ is a non decreasing function.

Lemma 1.7([13]): Let $(X, M, *)$ be a fuzzy metric space then M is continuous function on $X^2 \times (0, \infty)$.

Lemma 1.8([15]): Let $(X, M, *)$ be a fuzzy metric space if there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ then $x=y$.

Lemma 1.9([15]): Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition (fm6). If there exists a number $k \in (0, 1)$ such that $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $t > 0$ and $n \in \mathbb{N}$, then $\{y_n\}$ is a Cauchy sequence in X .

Proposition 1.10: In a fuzzy metric space $(X, M, *)$, if $a * a \geq a$ for all $a \in [0, 1]$ then $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Definition 1.11([3]): Two self maps A and S of a fuzzy metric space $(X, M, *)$ are called compatible if $\lim_{n \rightarrow \infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x$ for some x in X .

Definition 1.12 ([24]): Two self maps A and S of a fuzzy metric space $(X, M, *)$ are called weakly commuting if $M(ASx, SAx, t) \geq M(Ax, Sx, t)$ for all x in X and $t > 0$.

Definition 1.13 ([24]): Two self maps A and S of a fuzzy metric space $(X, M, *)$ are called R -weakly commuting if there exists $R > 0$ such that $M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R})$ for all x in X and $t > 0$.

Remark 1.14: Clearly, point wise R -weakly commuting implies weak commuting only when $R \leq 1$.

Remark 1.15: Compatible mappings are point wise R -weakly commuting but not conversely.

Definition 1.16([1]): Two self maps A and S of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if $\lim_{n \rightarrow \infty} ASx_n = Ax$ and $\lim_{n \rightarrow \infty} SAx_n = Sx$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x$ for some x in X .

Remark 1.17: If A and S are both continuous, then they are obviously reciprocally continuous but converse is not true. Moreover, in the setting of common fixed point theorems for pointwise R -weakly commuting maps satisfying contractive conditions, continuity of one of the mappings A or S implies their reciprocally continuity but not conversely.

The following theorem was proved by S. Kutukcu et. al. [12]

Theorem 1.18: Let $(X, M, *)$ be complete fuzzy metric space with $a*a \geq a$ for all $a \in [0, 1]$ and the condition (fm6). Let (A, S) and (B, T) be point wise R-weakly commuting pairs of self mappings of X such that

$$AX \subset X, BX \subset SX$$

There exists $k \in (0, 1)$ such that $M(Ax, By, kt) \geq N(x, y, t)$ where
 $N(x, y, t) = M(Sx, Ax, t) * M(Ty, By, t) * M(Sx, Ty, t) * M(Ty, Ax, \alpha t) * M(Sx, By, (2-\alpha)t)$ for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$.

If one of the mappings in compatible pair (A, S) or (B, T) is continuous then A, B, S and T have a unique common fixed point.

Our Results

In this section we extend the theorem 1.18 which generalize the results of [1], [18] and [12].

Definition 2.1[2]: Two self mappings A and S of a fuzzy metric space $(X, M, *)$ are called compatible of type (α) if $\lim_{n \rightarrow \infty} M(ASx_n, SSx_n, t) = 1$ and $\lim_{n \rightarrow \infty} M(SAx_n, AAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x$ for some $x \in X$.

We shall need the following lemma for proof of our main theorem.

Lemma 2.2 Let $(X, M, *)$ be a compatible fuzzy metric with $a*a \geq a$ for all $a \in [0, 1]$ and the condition (fm6). Let A, B, S, T, P and Q be mappings from X into itself such that

$$(2.2.1) \quad P(X) \subset AB(X), Q(X) \subset ST(X)$$

$$(2.2.2) \quad \text{There exist a } k \in (0, 1) \text{ such that } M(Px, Qy, kt) \geq N(x, y, t)$$

where $N(x, y, t) = M(STx, Px, t) * M(ABx, Qy, t) * M(STx, ABx, t) * M(ABx, Px, \alpha t) * M(STx, Qy, (2-\alpha)t)$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$ then the continuity of one of the mappings in compatible type (α) pair (P, ST) or (Q, AB) on $(X, M, *)$ implies their reciprocal continuity.

Proof: Suppose that (P, ST) are compatible of type (α) mappings and ST is continuous.

We claim that P and ST are reciprocally continuous.

Let $\{x_n\}$ be a sequence such that $Px_n \rightarrow z$ and $STx_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Since ST is continuous, we have

$$ST(STx_n) \rightarrow STz, ST(Px_n) \rightarrow STz \text{ as } n \rightarrow \infty.$$

Since (P, ST) is compatible of type (α) , we also have

$$\lim_{n \rightarrow \infty} M(PSTx_n, STSTx_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} M(STPx_n, PPx_n, t) = 1$$

$$\text{which gives } \lim_{n \rightarrow \infty} M(PSTx_n, STz, t) = 1 \text{ and } \lim_{n \rightarrow \infty} M(STz, PPx_n, t) = 1$$

that is $PST_{X_n} \rightarrow STz$ and $PP_{X_n} \rightarrow STz$ as $n \rightarrow \infty$.

Since $P(X) \subset AB(X)$, for each n there exists y_n in X such that $PST_{X_n} = AB_{y_n}$ thus we have

$$(2.2.3) \quad ST \ ST_{X_n} \rightarrow STz, \ STP_{X_n} \rightarrow STz, \ PP_{X_n} \rightarrow STz, \ PST_{X_n} = AB_{y_n} \rightarrow STz \text{ as } n \rightarrow \infty.$$

We claim that $Q_{y_n} \rightarrow STz$ as $n \rightarrow \infty$.

If not, there exists a sub sequence $\{Q_{y_m}\}$ of $\{Q_{y_n}\}$ such that for a given $t > 0$, there exists a number $\varepsilon > 0$ and positive integer n_0 such that for all $m > n_0$

$$M(Q_{y_m}, STz, t) \leq \varepsilon \text{ and } M(PST_{X_m}, Q_{y_m}, t) \leq \varepsilon.$$

Now from (2.2.2) and (2.2.3)

$$(2.2.4) \quad M(PST_{X_m}, Q_{y_m}, kt) \geq N(ST_{X_m}, y_m, t) \geq M(ST \ ST_{X_m}, PST_{X_m}, t) * M(AB_{y_m}, Q_{y_m}, t)$$

$$(2.2.5) \quad * M(ST \ ST_{X_m}, AB_{y_m}, t) * M(AB_{y_m}, PST_{X_m}, \alpha t) * M(ST \ ST_{X_m}, Q_{y_m}, (2-\alpha)t) \text{ letting } m \rightarrow \infty \text{ and } \alpha = 1 \text{ we get}$$

$$\begin{aligned} \lim_{m \rightarrow \infty} M(STz, Q_{y_m}, kt) &\geq \lim_{m \rightarrow \infty} (M(STz, STz, t) * M(STz, Q_{y_m}, t) * M(STz, STz, t) \\ &* M(STz, STz, t) * M(STz, Q_{y_m}, t)) \\ &\geq \lim_{m \rightarrow \infty} M(STz, Q_{y_m}, t) \end{aligned}$$

therefore from lemma 1.8, we have

$$(2.2.6) \quad \lim_{m \rightarrow \infty} Q_{y_m} = STz.$$

Now, from (2.2.6), (2.2.2), (2.2.3) we get

$$(2.2.7) \quad M(Pz, Q_{y_m}, kt) \geq N(z, y_m, t) = M(STz, Pz, t) * M(AB_{y_m}, Q_{y_m}, t) * M(STz, AB_{y_m}, t) * M(AB_{y_m}, Pz, \alpha t) * M(STz, Q_{y_m}, (2-\alpha)t) \text{ letting } m \rightarrow \infty \text{ and put } \alpha=1, \text{ we get}$$

$$(2.2.8) \quad M(Pz, STz, kt) \geq M(STz, Pz, t) * M(STz, STz, t) * M(STz, STz, t) * M(STz, Pz, t) * M(STz, STz, t)$$

Therefore from lemma 1.8, we have

$$(2.2.9) \quad STz = Pz,$$

thus $STP_{X_n} \rightarrow STz$ and $PST_{X_n} \rightarrow STz = Pz$ as $n \rightarrow \infty$, therefore ST and P are reciprocally continuous on X .

If the pair (Q, AB) is assumed to be compatible of type (α) and AB is continuous, the proof is similar.

Theorem 2.3: Let $(X, M, *)$ be complete fuzzy metric space with $a * a \geq a$ for all $a \in [0, 1]$ and the condition (fm6). Let A, B, S, T, P and Q be mappings from X into itself such that

- (2.3.1) $P(X) \subset AB(X), Q(X) \subset ST(X)$
 (2.3.2) The pairs $(A, B), (T, S), (P, S), (Q, A), (P, T), (Q, B)$ are commuting
 (2.3.3) The pairs (P, ST) and (Q, AB) are compatible of type (α) and one of P, AB, Q, ST is continuous.
 (2.3.4) There exists a number $k \in (0, 1)$ such that $M(Px, Qy, kt) \geq N(x, y, t)$

where $N(x, y, t) = M(STx, Px, t) * M(ABx, Qy, t) * M(STx, ABx, t)$

$$* M(ABx, Px, \alpha t) * M(STx, Qy, (2-\alpha)t)$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$

then A, S, P, T and B have a unique common fixed point in X .

Proof: $P(X) \subset AB(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$, since $Q(X) \subset ST(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Qx_1 = STx_2$.

Inductively we can find a sequence $\{y_n\}$ in X as follows

$$(2.3.5) \quad y_{2n} = Px_{2n} = ABx_{2n+1} \text{ and } y_{2n+1} = Qx_{2n+1} = STx_{2n+2} \text{ for } n=0, 1, 2, 3, \dots$$

By (2.3.4) for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$, we have

$$(2.3.6) \quad M(Px_{2n}, Qx_{2n+1}, kt) \geq M(STx_{2n}, Px_{2n}, t) * M(ABx_{2n+1}, Qx_{2n+1}, t) * M(STx_{2n}, ABx_{2n+1}, t)$$

$$(2.3.7) \quad * M(ABx_{2n+1}, Px_{2n}, (1-q)t) * M(STx_{2n}, Qx_{2n+1}, (1+q)t) \\ M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n-1}, y_{2n}, t) \\ * M(y_{2n}, y_{2n}, (1-q)t) * M(y_{2n-1}, y_{2n+1}, (1+q)t) \\ \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, qt)$$

Since t -norm $*$ is continuous, letting $q \rightarrow 1$, we have

$$\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, t) \\ \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t)$$

It follows that

$$(2.3.8) \quad M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t)$$

Similarly

$$(2.3.9) \quad M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$$

Therefore for all n even and odd, we have

$$(2.3.10) \quad M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) * M(y_n, y_{n+1}, t)$$

Consequently

$$(2.3.11) \quad M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-1}t) \\ \text{Repeated application of this inequality will imply that} \\ M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-1}t) \geq \dots \geq \dots \\ \dots \geq M(y_{n-1}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-m}t), m \in \mathbb{N}$$

Since $M(y_n, y_{n+1}, k^{-m}t) \rightarrow 1$ as $m \rightarrow \infty$, it follows that

$$(2.3.12) \quad M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, k^{-1}t) \text{ for all } n \in \mathbb{N}.$$

Consequently $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $n \in \mathbb{N}$.

Therefore, by the lemma 1.9, $\{y_n\}$ is a Cauchy sequence in X . Since X is complete, $\{y_n\}$ converges to a point $z \in X$. Since $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are sub sequences of $\{y_n\}$, they also converge to the point z .

$$(2.3.13) \quad Px_{2n}, Qx_{2n+1}, ABx_{2n+1}, STx_{2n+1} \rightarrow z \text{ as } n \rightarrow \infty.$$

Assume P is continuous.

$$(2.3.14) \quad PPx_{2n} = PABx_{2n+1} \rightarrow Pz \text{ and } PQx_{2n+1} = PSTx_{2n+2} \rightarrow Pz \text{ as } n \rightarrow \infty.$$

Since (P, ST) is compatible of type (α) ,

$$(2.3.15) \quad \lim_{n \rightarrow \infty} M(PSTx_{2n}, STSTx_{2n}, t) = 1 \text{ and } \lim_{n \rightarrow \infty} M(STPx_{2n}, PPx_{2n}, t) = 1.$$

Since (P, ST) is compatible of type (α) and P is continuous, by lemma 2.2 P, ST are reciprocally continuous, so

$$(2.3.16) \quad PSTx_{2n} \rightarrow Pz, STPx_{2n} \rightarrow STz \text{ as } n \rightarrow \infty.$$

In view of (2.3.15) and (2.3.16) we get

$$\lim_{n \rightarrow \infty} M(Pz, STSTx_{2n}, t) = 1 \text{ and } \lim_{n \rightarrow \infty} M(STz, PPx_{2n}, t) = 1 \text{ thus}$$

$$(2.3.17) \quad STSTx_{2n} \rightarrow Pz \text{ and } PPx_{2n} \rightarrow STz \text{ as } n \rightarrow \infty$$

From (2.3.17) and (2.3.14) we get

$$(2.3.18) \quad Pz = STz$$

Since $P(X) \subset AB(X)$, there exists a point w in X such that $Pz = ABw$.

Using (2.3.4) with $\alpha=1$ and (2.3.18), we get

$$(2.3.19) \quad M(Pz, Qw, kt) \geq N(z, w, t) = M(STz, Pz, t) * M(ABw, Qw, t) * M(STz, ABw, t)$$

$$\begin{aligned} & * M(ABw, Pz, \alpha t) * M(STz, Qw, (2-\alpha)t) \\ & \geq M(Pz, Pz, t) * M(Pz, Qw, t) * M(Pz, Pz, t) * M(Pz, Pz, t) * M(Pz, Qw, t) \\ & M(Pz, Qw, kt) \geq M(Pz, Qw, t) \text{ which shows } Pz = Qw \text{ thus} \end{aligned}$$

$$(2.3.20) \quad Pz = STz = Qw = ABw.$$

Now, from (2.3.13) and (2.3.4) with $\alpha=1$ we get

$$(2.3.21) \quad \begin{aligned} M(Pz, z, kt) &= \lim_{n \rightarrow \infty} M(Pz, Qx_{2n+1}, kt) \\ &\geq \lim_{n \rightarrow \infty} N(z, x_{2n+1}, t) \\ &\geq \lim_{n \rightarrow \infty} \{M(STz, Pz, t) * M(ABx_{2n+1}, Qx_{2n+1}, t) * M(STz, ABx_{2n+1}, t) \\ & * M(ABx_{2n+1}, Pz, t) * M(STz, Qx_{2n+1}, t)\} \\ &\geq M(Pz, Pz, t) * M(z, z, t) * M(Pz, z, t) * M(z, Pz, t) * M(Pz, z, t) \\ &M(Pz, z, kt) \geq M(Pz, z, t) \text{ which shows } z = Pz \text{ thus} \end{aligned}$$

$$(2.3.22) \quad Pz=z=STz=Qw=ABw.$$

Taking $w_n=w$ for all $n \geq 1$, it follows that $Qw_n \rightarrow Qw=z$ and $ABw_n \rightarrow ABw=z$ as $n \rightarrow \infty$

Since (Q, AB) is compatible mappings of type (α)

$$\lim_{n \rightarrow \infty} M(QQw_n, ABQw_n, t) = 1$$

$$M(QQw, ABQw, t) = 1$$

$M(Qz, ABz, t) = 1$ which implies $Qz=ABz$ thus

$$(2.3.23) \quad Qz = ABz.$$

Now we show that z is the fixed point of Q

$$(2.3.24) \quad \begin{aligned} M(z, Qz, kt) &= \lim_{n \rightarrow \infty} M(Px_{2n}, Qz, kt) \geq \lim_{n \rightarrow \infty} N(x_{2n}, z, t) \\ &\geq \lim_{n \rightarrow \infty} \{ M(STx_{2n}, Px_{2n}, t) * M(ABz, Qz, t) * M(STx_{2n}, ABz, t) * M(ABz, Px_{2n}, \\ &\alpha t) * M(STx_{2n}, Qz, (2-\alpha)t) \} \end{aligned}$$

letting $\alpha=1$

$$\geq M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t) * M(Qz, z, t) * M(z, Qz, t)$$

$$\geq M(z, Qz, t) \text{ which shows } z=Qz \text{ thus}$$

$$(2.3.25) \quad z = Pz = Qz = ABz = STz.$$

Now we show that $z=Tz$ by putting $x=Tz, y=x_{2n+1}$ with $\alpha=1$ in (2.3.4) and using (2.3.2), (2.3.25)

$$(2.3.26) \quad \begin{aligned} M(PTz, Qx_{2n+1}, kt) &\geq N(Tz, x_{2n+1}, t) \\ &\geq M(STTz, PTz, t) * M(ABx_{2n+1}, Qx_{2n+1}, t) * M(STTz, ABx_{2n+1}, t) \\ &* M(ABx_{2n+1}, PTz, t) * M(STTz, Qx_{2n+1}, t) \end{aligned}$$

letting limit as $n \rightarrow \infty$

$$M(Tz, z, kt) \geq M(Tz, Tz, t) * M(z, z, t) * M(Tz, z, t) * M(z, Tz, t) * M(Tz, z, t)$$

$$\geq M(Tz, z, t) \text{ which shows } Tz=z.$$

$$(2.3.27) \quad Tz = z.$$

Since $STz=z$ therefore $Sz=z$.

Finally we have to show that $Bz=z$.

By putting $x=z, y=Bz$ with $\alpha=1$ in (2.3.4) and using (2.3.27), (2.3.25)

$$(2.3.28) \quad \begin{aligned} M(Pz, QBz, kt) &\geq N(z, Bz, t) \\ &\geq M(STz, Pz, t) * M(ABBz, QBz, t) * M(STz, ABBz, t) \\ &* M(ABBz, Pz, t) * M(STz, QABz, t) \\ &\geq M(z, z, t) * M(Bz, Bz, t) * M(z, Bz, t) * M(Bz, z, t) * M(z, z, t) \\ &\geq M(z, Bz, t) \end{aligned}$$

which implies $M(z, Bz, kt) \geq M(z, Bz, t)$

therefore $z=Bz$.

since $ABz=z$ therefore $Az=z$.

By combining the above results, we have $Az=Bz=Sz=Tz=Pz=Qz=z$ that is z is the common fixed point of A, B, S, T, P and Q .

For uniqueness, let $w(w \neq z)$ be another common fixed point of A, B, S, T, P and Q with $\alpha=1$.

Then by(2.3.4), we get $M(Pz, Qw, kt) \geq M(STz, Pz, t) * M(ABw, Qw, t) * M(STz, ABw, t)$

$$* M(ABw, Pz, t) * M(STz, Qw, t) \\ \geq M(z, z, t) * M(w, w, t) * M(z, w, t) * M(w, z, t) * M(z, w, t)$$

$$M(z, w, kt) \geq M(z, w, t),$$

therefore $z=w$, z is the unique fixed point of A, B, P, Q, S and T .

If we put $B=T=I_x$ (the identity map on X) in the theorem 2.3 we have the following.

Corollary 2.4: Let $(X, M, *)$ be a complete fuzzy metric space with $a * a \geq a$ for all $a \in [0, 1]$ and the condition (fm6), let A, S, T be mapping from X into itself such that

$$(2.4.1) \quad P(X) \subset A(X), Q(X) \subset S(X)$$

(2.4.2) The pairs (P, S) and (Q, A) are compatible of type (α) and one of P, A, Q and S is continuous.

(2.4.3) There exists a number $k \in (0, 1)$ such that

$$M(Px, Qy, kt) \geq M(Sx, Px, t) * M(Ay, Qy, t) * M(Sx, Ay, t) * M(Ay, Px, \alpha t)$$

$* M(Sx, Qy, (2-\alpha)t)$ for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$ then A, S, P and Q have a unique common fixed point in X .

If we put $P=Q, B=T=I_x$ in the theorem 2.3 we have the following.

Corollary 2.5: Let $(X, M, *)$ be a complete fuzzy metric space with $a * a \geq a$ for all $a \in [0, 1]$ and the condition (fm6). Let A, S, T be mappings from X into itself such that

$$(2.5.1) \quad P(X) \subset A(X), P(X) \subset S(X)$$

(2.5.2) The pairs $(P, A), (P, S)$ are compatible of type (α) and one of P, A, S is continuous.

$$(2.5.3) \quad M(Px, Py, kt) \geq M(Sx, Px, t) * M(Ay, Py, t) * M(Sx, Ay, t) * M(Ay, Px, \alpha t) * M(Sx, Py, (2-\alpha)t)$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$ then P, S, A have a unique common fixed point in X .

If we put $A=S$ and $B=T=I_x$ in the theorem 2.3 we have the following.

Corollary 2.6: Let $(X, M, *)$ be complete fuzzy metric space with $a * a \geq a$ for all $a \in [0, 1]$ and the condition (fm6). Let P, Q and S be mappings from X to itself such that

$$(2.6.1) \quad P(X) \subset S(X), Q(X) \subset S(X)$$

(2.6.2) The pairs (P, S) or (Q, S) is compatible of type (α) and one of P, S, Q is continuous

$$(2.6.3) \quad M(Px, Qy, kt) \geq M(Sx, Px, t) * M(Sy, Qy, t) * M(Sx, Qy, t) * M(Sy, Px, \alpha t) * M(Sx, Qy, (2-\alpha)t)$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$ then P, Q, S have a unique common fixed point

in X .

If we put $P=Q$, $A=S$ and $B=T=I_x$ in the theorem 2.3 we have the following.

Corollary 2.7: Let $(X, M, *)$ be complete fuzzy metric space with $a*a \geq a$ for all $a \in [0, 1]$ and the condition (fm6). Let P, S be compatible mappings of type (α) of self maps such that $P(X) \subset S(X)$, and there exists a constant $k \in (0, 1)$ such that

$$M(Px, Py, kt) \geq M(Sx, Px, t) * M(Sy, Py, t) * M(Sx, Sy, t) * M(Sy, Px, \alpha t) * M(Sx, Py, (2-\alpha)t)$$

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$ then P and S have a unique common fixed point in X .

Remark 2.8: Since $a*b = \min\{a, b\}$ then the condition (2.3.4) in the theorem 2.3 becomes

$$M(Px, Qy, kt) \geq \min\{M(STx, Px, t) * M(ABy, Qy, t) * M(STx, ABx, t) * M(ABx, Px, \alpha t) * M(STx, Qy, (2-\alpha)t)\}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$.

Now, we prove the theorem 2.3 from complete fuzzy metric space to complete metric space.

Theorem 2.9: Let A, B, S, T, P and Q be self mappings of a complete metric space (X, d) suppose that the pairs (P, AB) and (Q, ST) are compatible mappings of type (α) such that $P(X) \subset S(X)$, $Q(X) \subset S(X)$ and also the pairs (A, B) , (S, T) , (P, S) , (P, T) , (Q, A) , (B, Q) are commuting, one of P, Q, AB, ST are continuous. If there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$

$$d(Px, Qy) \leq k \max\{d(STx, Px), d(ABx, Qy), d(STx, ABx), \frac{1}{2}[d(ABx, Px) + d(STx, Qy)]\}$$

then A, B, S, T, P and Q have a unique common fixed point in X .

Proof: The proof follows from theorem 2.3, by considering the induced fuzzy metric space $(X, M, *)$ where $a*b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$.

This result also generalizes the results of Pant and Jha[18] and Balasubramaniam et.al.[1],

Kutukcu et.al.[12] for complete metric space in the above said sense.

References

- [1] P. Balasubramaniam, S. MuraliSankar, R.P.Pant, Common fixed points of four mappings in a fuzzy metric space, J.Fuzzy Math. 10(2)(2002), 379-384.
- [2] Y.J.Cho, Fixed Points in fuzzy metric spaces, J.Fuzzy Maths. 5(1997), 949-962.

- [3] Y.J.Cho, H.K.Pathak, S.M. Kang, J.S. Jung, Common fixed points of compatible maps of type (β) on fuzzy metric spaces, *Fuzzy Sets and Systems* 93(1998), 99-111.
- [4] A. George, P.Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64(1994), 395-399.
- [5] M.Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems* 27(1988), 385-399.
- [6] O.Hadzic, Common fixed point theorems for families of mapping in complete metric space, *Math.Japon.* 29(1984), 127-134.
- [7] G. Jungck, Compatible mappings and common fixed Points(2), *Internat.J.Math.Math. Sci.*(1988), 285-288.
- [8] I.Kramosil and J.Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, 11(1975), 326-334.
- [9] S.Kutukcu, A fixed point theorem in menger spaces, *Int.Math.F.1*(32)(2006), 1543-1554.
- [10] S.Kutukcu, C.Yildiz, A. Juna, On common fixed points in Menger probabilistic metric spaces, *Int.J.Contemp.Math.Sci.*2(8)(2007), 383-391.
- [11] S.Kutukcu, A fixed point theorem for contraction type mappings in Menger Spaces.*Am.J.Appl.Sci.*4(6)(2007), 371-373.
- [12] S.Kutukcu, S.Sharma, H.Tokgoz, A fixed point theorem in fuzzy metric spaces. *Int.J.Math.Analysis*, 1(18)(2007), 861-872.
- [13] J. R. Lopez, S. Romaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems*, 147(2004), 273-283.
- [14] S.N.Mishra, Common fixed points of Compatible mappings in PM-Spaces, *Math.Japon.*36(1991), 283-289.
- [15] S.N.Mishra, N.Sharma and S.L.Singh, Common fixed Points of maps on fuzzy metric spaces, *Internet.J.Math & Math.Sci*, 17(1994), 253-258.
- [16] R.P.Pant, Common fixed points of four mappings, *Bull.Cal.Math.Soc.*90(1998), 281-286.
- [17] R.P.Pant, Common fixed point theorems for contractive maps, *J.Math.Anal.Appl.*226(1998), 251-258.
- [18] R.P.Pant, K. Jha, Aremarkon, Common fixed points of four mappings in a fuzzy metric space, *J.Fuzzy Math.*12(2)(2004), 433-437.
- [19] B.E.Rhoades, Contractive definitions and continuity, *Contemporary Math.*72(1988), 233-245.
- [20] S.Sharma, Common fixed point theorems in fuzzy metric spaces, *Fuzzy Sets and Systems* 127(2002), 345-352.
- [21] S.Sharma, B.Deshpande, Discontinuity and weak compatibility in fixed point consideration on non-complete fuzzy metric spaces, *J.Fuzzy Math.*11(2)(2003), 671-686.
- [22] S.Sharma, B.Deshpande, Compatible multivalued mappings satisfying an implicit relation, *Southeast Asian. Bull.Math.*30(2006), 535-540.
- [23] G.Song, Comments on A Common fixed point theorem in a fuzzy metric space, *Fuzzy sets and Systems* 135 (2003) 409-413

- [24] R.Vasuki, Common fixed points for R-weakly commuting maps in fuzzy metric spaces, *Indian J.Pure.Appl.Math.*30(1999), 419-423.
- [25] R. Vasuki and P. Veeramani, Fixed point theorem and Cauchy sequences in fuzzy metric spaces, *Fuzzy Sets and Systems*, 135(2003), 415-417.
- [26] L.A.Zadeh, *Fuzzy Sets, Information and Control*, 8(1965) 338-353.