A New Structures of Fuzzy G–Modular Distributive Lattices.

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Abstract

The main goal of this paper is to study the finite groups whose lattices of fuzzy G-modular are distributive. We obtain a characterization of these lattices which is similar to a well known result of lattice theory.

Keywords: Fuzzy lattices, level set, G-modular, Fuzzy G-modular lattices, fuzzy filter, fuzzy ideal, lattice homomorphism.

Introduction

The theory of fuzzy sets which was introduced by L. A. Zadeh [20] is applied to many mathematical branches. Rosenfeld [18] inspired the fuzzification of algebraic structures and introduced the notion of fuzzy sub groups. P. Das [7] characterized fuzzy sub groups by their level sub groups. In [13] Liu applied the concept of fuzzy sets to the theory of rings and introduced and examined the notion of a fuzzy ideal of a ring. The study of fuzzy sub module was introduced by pan and Golan in [9]. Also Pan [17] studied the fuzzy finitely generated modules and fuzzy quotient modules. Later Katsaras and D.B. Liu introduced the concept of fuzzy vector spaces and fuzzy topological vector spaces. The formation of a lattice of sub modules of a module is well known features in classical algebra. However the same has not been explored in fuzzy setting. In order to initiate such studies the concept of fuzzy sub-module generated by an arbitrary fuzzy set is formulate in this note. Using this concept S.K. Bambri and Pratibakumar [6] introduced Lattice of fuzzy sub modules and established an embedding of the lattice of all sub module of a module M into the lattice of fuzzy
sub module M. Marudai and V. Rajendra in [15] introduced fuzzy G-modules on a fuzzy sub lattice $L$ and characterized it in different aspects.

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like.

This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting. One of the structures that are most extensively used and discussed in mathematics and its applications is lattice theory. As it is well known, it is considered as a relational ordered structure, on one hand, and as an algebra, on the other hand. B. Davvaz & O. Kazanci in [8] introduces a new kind of fuzzy sub lattice (ideal, filter) of a lattice.

Fuzzy algebraic sub structures are important when viewed from a lattice theoretic point of view. N. Ajmal and K.V. Thomas initiated such types of studies in the year 1994 [3]. It was later independently established by Ajmal [1] that the set of all fuzzy normal sub groups of a group constitute a sub lattice of a the lattice of all fuzzy sub groups of a given group and is modular. Tom Head also discussed the modularity of a set of fuzzy sub groups of a group in his pioneering paper [10], wherein he formulated the well-known meta-theorem. However the assertion that the set of all fuzzy normal sub groups constitute a modular lattice, follows from sub direct product theorem, is false in his proof. Tom Head used the fact that the rep function commutes with the set product of two fuzzy normal sub groups. In fact the join of two fuzzy normal sub groups is their set product if and only if they have the same tip. Therefore what follows from the sub direct product theorem is the fact that the set of all fuzzy normal sub groups with the same tip is modular.

In a series of papers [1, 2, 3] various sub lattice of the lattice $\mathcal{L}$ of all fuzzy sub groups of a group $G$ are constructed and examined. On the other hand, for the set of all fuzzy ideals of a ring $R$, the only notable attempt so far, has been made in the year 1995 [3]. Wherein it has been shown that the set of all fuzzy ideals $I(R)$ constitute a lattice under the ordering of fuzzy set inclusion. Moreover, the construction of a particular sub lattice of $I(R)$, that is, the sub lattice $I_{st}(R)$ of all fuzzy ideals, each of them having sup property and the same tip “t” is presented in [1]. In the same paper, they showed that, that lattice is to be modular. More than the modularity of this lattice, its construction, which arises from a property of functions is important.

Even after the emergence of meta-theorem in 1995, a direct proof of modularity of the lattice of fuzzy ideals of a ring was published by Q. Zhang [22] in 2002. The proof of modularity in this paper is unnecessarily long, complicated and tedious and the author tries to demonstrate the utility of nested sets to establish modularity. However, the fact is, that the underlying structure of the proof is exactly same as that of modularity of the lattice of fuzzy normal sub groups given by Ajmal [1]. Whose work the author very generously refers to in his paper [1]. Another attempt for establishing the modularity of fuzzy ideals of a ring has been made by Q. Zhang jointly with G. Meng in the year 2000 [21]. Where in the authors prove that the sub lattice $I_{t}(R)$ of all fuzzy ideals with the same tip $t$ of a ring $R$ is modular. This proof is also long and is
again almost similar to the proof given by Ajmal and Thomas [3], wherein they prove that the sub lattice $L_{\text{fin}}$ of all fuzzy normal sub groups with finite range sets and some time tip is modular.

Q. Zhang and Meng [21] considered the sub property to be an assumption and established the more general result that “the lattice $l_t(R)$ of all fuzzy ideals of a ring $R$ with the same tip “$t$” is modular. On the other hand in [14] the authors have arrived at the false result that The lattice $I(R)$ is distributive. In fact the lattice $I(R)$ of ideals of a ring $R$ is not distributive and has an obvious embedding in $l(R)$. The corrected result has already appeared in several papers [12, 21]. I Jahan in [11], by constructing and employing the technique of strong level subsets, he proved that the lattice $(R)$ of all ideals of a ring $R$ is modular. This proof of the modularity of $l(R)$ is different from the Ajmal’s proof of modularity of the lattice of fuzzy normal sub groups of group appeared in [1]. K.C.Gupta and S. Roy in introduce the indirect proof of the above result that the modularity of the quasi hamiltonian fuzzy sub groups. Hence N. Ajmal and K.V. Thomas in [3] initiated a discussion on the aspect of modularity of the set of fuzzy normal sub groups. A special class of fuzzy normal sub groups of group has been shown to constitute a modular sub lattice of its fuzzy sub group lattice. They first established that the sub lattice $(G)$ is modular. Moreover using the same technique in [4], they demonstrate that whenever the set $(G)$ of all fuzzy quasi normal sub groups of a given group forms a lattice, the lattice is modular. B. Yuan and Wu. Wangming in [5], introduced fuzzy ideals on a distributive lattice.

In this paper, we study the characterization of fuzzy G-modular distributive lattice such as ring sums, lattice homomorphism and upper level sets.

**Preliminaries**

**Definition 2.1** A non empty set $L$ together with two binary operations $V$ and $\Lambda$ on $L$ is called a lattice if it satisfies the following identities.

\[
\begin{align*}
L1 & : \quad (a) \quad x \lor y \leq y \lor x \\
& \quad x \land y \leq y \land x \\
L2 & : \quad (a) \quad x \lor (y \lor z) \leq (x \lor y) \lor z \\
& \quad x \land (y \land z) \leq (x \land y) \land z \\
L3 & : \quad (a) \quad x \lor x \leq x \\
& \quad x \land x \leq x \\
L4 & : \quad (a) \quad x \leq x \lor (x \land y) \\
& \quad x \leq x \land (x \lor y)
\end{align*}
\]

The operation $\Lambda$ is called meet and the operation $V$ is called join. Let $L$ be the set of proposition and let $V$ denote the connective “or” and $\Lambda$ denote the connective “and”. Then L1 to L4 are well known properties from prepositional logic.
**Definition 2.2** A distributive lattice is a lattice which satisfies either of the distributive laws.

\[ D1 \quad x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \]

\[ D2 \quad x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z) \]

One can see a lattice \( L \) satisfies \( D1 \) if and only if satisfies \( D2 \).

**Definition 2.3** Let \( G \) be a group and \( L \) be a vector distributive lattice then \( L \) is called \( G \)-modular if for \( a \in G \) and \( l \in L \), there exist a product \( a.l \in L \) satisfies the following axioms.

(i) \( e. l = l \), for all \( l \in L \)

(ii) \((a. h). l = a (h. l)\)

(iii)\( a. (k_1 m_1 + k_2 m_2) = k_1 (am_1) + k_2(am_2) \) for all \( k_1, k_2 \in K \) and \( m_1, m_2 \in M \).

**Definition 2.4** A mapping \( \mu : X \rightarrow [0, 1] \) where \( X \) is an arbitrary non-empty set and is called a fuzzy set in \( X \).

**Definition 2.5** Let \( X \) be a fuzzy set and \( \mu : X \rightarrow G \) be a lattice ordered group of \( X \). Then \( \mu \) is called fuzzy lattice ordered group (FLG) if

(i) \( \mu(x + y) \geq \min \{\mu(x), \mu(y)\} \)

(ii) \( \mu(–x) \geq \mu(x) \)

(iii)\( \mu(0) = 1 \), for all \( x, y \in G \).

**Definition 2.6** Let \( \mu \) be a fuzzy lattice ordered group of \( G \) and \( \mu : X \rightarrow G \). Then \( \mu \) is called fuzzy lattice if

(i) \( \mu(x + y) \geq \min \{\mu(x), \mu(y)\} \)

(ii) \( \mu(–x) \geq \mu(x) \)

(iii)\( \mu(x \vee y) \geq \min \{\mu(x), \mu(y)\} \)

(iv)\( \mu(x \wedge y) \geq \min \{\mu(x), \mu(y)\} \), for all \( x, y \in G \).

**Definition 2.7** Let \( X \) be any group and \( L \) be a vector distributive lattice extended real valued functions on \( X \). If \( \mu \) is called a fuzzy \( G \)-modular lattice on \( L \) then it satisfies the following conditions.

(i) \( \mu(ax + by) \geq T\{\mu(x), \mu(y)\} \)

(ii) \( \mu(gx) \geq \mu(x) \)

(iii)\( \mu(x \vee y) \wedge \mu(x \wedge y) \geq T\{\mu(x), \mu(y)\} \), for all \( x, y \in L \).

**Definition 2.8:** A fuzzy sub set \( \mu \) is called monotonic if

(i) \( \mu(x) \leq \mu(y) \) whenever \( x \leq y \)

**Definition 2.9:** Let \( \mu \) be any fuzzy sub set of \( X \). Then the set

(i) \( \mu_t = \{x \in X / \mu(x) \geq t, t \in [0, 1]\} \) is called a level sub set of \( \mu \).
Definition 2.10: (i) A monotonic fuzzy G-modular lattice is called a fuzzy filter of L.
(ii) A anti-monotonic fuzzy G-modular lattice is called a fuzzy ideal of L.
(iii) A fuzzy subset $\mu$ is called a fuzzy filter if and only if $\mu(x \land y) = \mu(x) \land \mu(y)$, for all $x, y \in L$.
(iv) A fuzzy subset $\mu$ is called a fuzzy ideal if and only if $\mu(x \lor y) = \mu(x) \land \mu(y)$, for all $x, y \in L$.

Definition 2.11: (i) A fuzzy filter is prime if $\mu(x \lor y) \leq S\{\mu(x), \mu(y)\}$
(ii) A fuzzy ideal is prime if $\mu(x \land y) \leq S\{\mu(x), \mu(y)\}$, holds for all $x, y \in L$.

Definition 2.12: A fuzzy G-modular lattice A is modular if and only if $A(x, z) \geq 0 \Rightarrow x \lor (y \land z) = (x \lor y) \land z$ for all $x, y$ and $z \in X$.

Definition 2.13: Let $\mu$ be a fuzzy G-modular lattice, if $\mu$ is distributive then
(i) $x \land (y \lor z) = (x \land y) \lor (x \land y)$ and
(ii) $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ for all $x, y$ and $z \in X$.

Definition 2.14: Let L and $L'$ be lattices. A mapping $f: L \rightarrow L'$ is said to be lattice homomorphism if
(i) $f(x + y) = f(x) + f(y)$ and
(ii) $f(xy) = f(x). f(y)$, for all $x, y \in L$.

Definition 2.15: We say that a fuzzy G-modular distributive lattice $M = (L, \mu_M)$ in L satisfies the imaginable property if
$\text{I}_{\text{in}}(\mu_M) \subseteq \Delta T$

Definition 2.16: Let $\lambda$ and $\mu$ be two fuzzy G-modular distributive lattice of L then the sum of $\lambda$ and $\mu$ are denoted by $\lambda + \mu$ and is defined as
$(\lambda + \mu)(z) = \sup \{\min \{\lambda(x), \mu(y)\}\}$ where $x, y \in L$.
$z = x + y$

Clearly $\lambda + \mu$ is a fuzzy G-modular distributive lattice in L.

Definition 2.17: Let $\lambda$ and $\mu$ be two fuzzy G-modular distributive lattice of L, Then the ring sum of $\lambda$ and $\mu$ is defined as
$(\lambda \oplus \mu)(z) = \sup \{{(\mu_1 + \mu_2)(z) / x = f^{-1}(\lambda), y = f^{-1}(\mu)}\}$
$Z \in \lambda \oplus \mu$
for all $x, y$ and $z \in L$. 
Characaterization of Fuzzy G-Modular Distributive Lattices
Throughout this paper G be a finite groups and L = <L, +, · > denotes a lattice.
That is of maps from L into <[0, 1], V, Λ>, where [0, 1] is the set of reals between 0 and 1.
and
(i) \( x \odot y = \max (x, y) \), if \( y \leq x \) then \( \mu(y) \geq \mu(x) \)
(ii) \( x \circ y = \min (x, y) \), if \( x \leq y \) then \( \mu(x) \geq \mu(y) \)

Proposition 3.1: Let M be a G-modular lattice and N be a fuzzy G-sub modular lattice of M has a fuzzy G-modular distributive lattice then M/N and N has fuzzy G-modular distributive lattice. (FGMDL).

Proof: Let \( \mu \) be a fuzzy sub modular lattice of M then \( V: \mu_N \) is a fuzzy G-modular lattice on N.

Define \( \mu : \frac{M}{N} \rightarrow [0, 1] \) by
\[
\mu (x + N) = \mu (x), \text{ for all } x \in G \text{ and } n \in \frac{M}{N}
\]

(FGMDL1) \( \mu (a(x + N) + b(y + N)) = \mu ((ax + by) + N) = \mu (ax + by) \)
\[
\geq T \{ \mu (x) + \mu (y) \}
\geq T \{ \mu (x + N) + \mu (y + N) \}
\]

(FGMDL2) \( \mu (g(x + N)) = \mu (gx + N) = \mu (gx) \)
\[
\geq \mu (x) \geq \mu (x + N)
\]

(FGMDL3)
\[
(\mu (x \odot y) \Lambda \mu (x \circ y)) + N = \min \{ \mu ((x \odot y) + N, \mu (x \circ y) + N) \}
\geq \min \{ \{ \min \{ (\mu(x), \mu(y)) + N \}, \}
\{ \min \{ (\mu(x), \mu(y)) + N \} \}
\geq T \{ \mu (x) + N, \mu (y) + N \}
\geq T \{ \mu (x + N), \mu (y + N) \}
\]

(\text{since } (\mu (x + N) = \mu (x)) \text{ and } (\mu (y + N) = \mu (y))
\geq T \{ \mu (x), \mu (y) \}, \text{ for all } x, y \in G

Therefore \( \frac{M}{N} \) and N are fuzzy G-modular distributive lattice.

Proposition 3.2: Let \( f: L \rightarrow L^* \) be a G-modular distributive lattice homomorphism.
Where \( L \) and \( L^* \) are G-modular distributive lattices. If \( V \) is a fuzzy G–modular distributive lattice on \( L^* \) then \( f^{-1}(V) \) is a fuzzy G-modular distributive lattice on \( L \).

**Proof:** Since \( V \) is fuzzy G-modular distributive lattice on \( L^* \) and \( f: L \to L^* \) be a G-modular lattice homomorphism. For any \( a, b \in K \) and \( x, y \in L \), we have

\[
\text{(FGMDL1)} \quad f^{-1}(V)(ax + by) = V(f(ax + by)) \\
= V(f(ax) + f(by)) \\
\geq V\{f(x) + f(y)\} \\
\geq T\{Vf(x), Vf(y)\} \\
\geq T\{f^{-1}(V)(x), f^{-1}(V)(y)\}
\]

\[
\text{(FGMDL2)} \quad f^{-1}(V)(gm) = Vf(gm) \\
\geq Vf(m) \\
\geq f^{-1}(V)(m)
\]

\[
\text{(FGMDL3)} \quad f^{-1}(V)(x \lor y) \land f^{-1}(V)(x \land y) = \min \{f^{-1}(V)((x \lor y), f^{-1}(V)(x \land y)\} \\
= \min \{Vf(x \lor y), Vf(x \land y)\} \\
= \min \{V(f(x) + f(y)), V(f(x) \cdot f(y))\} \\
\text{(since } V \text{ is homomorphism} \\
\geq \min \{T\{Vf(x), Vf(y)\}, T\{Vf(x), Vf(y)\}\} \\
\geq \min \{Vf(x), Vf(y)\} \\
\geq T\{f^{-1}(V)(x), f^{-1}(V)(y)\}
\]

therefore \( f^{-1}(V) \) is a fuzzy G-modular distributive lattice on \( L \).

**Preposition 3.3:** Let \( T \) be a t-norm, then every imaginable fuzzy G-modular distributive lattice of \( L \) is fuzzy G-modular distributive lattice.

**Proof:** Let \( M = (L, \mu_M) \) be an imaginable fuzzy G-modular distributive lattice of \( L \) under T-norms. Then

(i) \( \mu_M(ax + by) \geq T\{\mu_M(x), \mu_M(y)\} \)

(ii) \( \mu_M(gm) \geq \mu_M(m) \)

(iii) \( \mu_M(x \lor y) \land \mu_M(x \land y) \geq T\{\mu_M(x), \mu_M(y)\} \) for all \( a, b \in G, x, \) and \( y \) and \( m \in M \).

We have

\[
\min \{\mu_M(x), \mu_M(y)\} = T\{\min \{\mu_M(x), \mu_M(y)\}, \min \{\mu_M(x), \mu_M(y)\}\} \\
\leq T\{\mu_M(x), \mu_M(y)\} \\
\leq \min \{\mu_M(x), \mu_M(y)\}
\]
It follows that $\mu_M (ax + by) \geq T \{\mu_M (x), \mu_M (y)\}$

Therefore $M = (L, \mu_M)$ is an imaginable fuzzy G-modular distributive lattice of $L$.

**Proposition:** 3.4 If $L$ is a complete lattice then the intersection of a family of a fuzzy G-modular distributive lattice is a fuzzy G-modular distributive lattice.

**Proof:** Let $\{ M_j : j \in J \}$ be a family of fuzzy G-modular distributive lattice and let $M = \bigcap_{j \in J} M_j$

We have $\mu_M (x) = \inf_{j \in J} \mu_{M_j} (x)$

and $\mu_M (y) = \inf_{j \in J} \mu_{M_j} (y)$

(FGMDL1) $\mu_M (ax + by) = \inf_{j \in J} \mu_{M_j} (ax + by)$

$\geq \inf_{j \in J} \{\min \{\mu_{M_j} (x), \mu_{M_j} (y)\}\}$

$\geq \min \{\inf_{j \in J} \mu_{M_j} (x), \inf_{j \in J} \mu_{M_j} (y)\}$

$\geq T \{\mu_M (x), \mu_M (y)\}$

(FGMDL2) $\mu_M (gx) = \inf_{j \in J} \mu_{M_j} (gx)$

$\geq \inf_{j \in J} \mu_{M_j} (x)$

$\geq \mu_M (x)$

(FGMDL3)

$\mu_M (x \lor y) \land \mu_M (x \land y) = \min \{\mu_M ((x \lor y), \mu_M (x \land y)\}$

$= \min \{\inf_{j \in J} \mu_{M_j} (x \lor y), \inf_{j \in J} \mu_{M_j} (x \land y)\}$

$= \min \{\inf_{j \in J} T \{\mu_{M_j} (x), \mu_{M_j} (y)\}\}$

$\inf_{j \in J} T \{\mu_{M_j} (x), \mu_{M_j} (y)\} \geq \min \{\inf_{j \in J} \mu_{M_j} (x), \inf_{j \in J} \mu_{M_j} (y)\}$

$\geq T \{\mu_M (x), \mu_M (y)\}\text{, for all } x, y \in L.$

Therefore $\bigcap_{j \in J} \mu_{M_j}$ is a fuzzy G-modular distributive lattice.

**Proposition:** 3.5 Any finite dimensional G-modular distributive lattice over a sub lattice $L$ is a fuzzy G-modular distributive lattice over $L$. 
Proof: Let $\mu: M \to [0, 1]$ be a map and $M$ be a finite $n$-dimensional fuzzy $G$-modular distributive lattice over $L$.

For all $\alpha, \beta \in L$ and $x, y \in M$
Such that $\mu^n(\alpha x) = \mu(n(\alpha x)) = \mu^n(x)$

$$(FGMDL1) \quad \mu^n(\alpha x + \beta y) = \mu(n(\alpha x + \beta y))$$
$$= n \mu(\alpha x + \beta y)$$
$$\geq n T \{ \mu(\alpha x) + \mu(\beta y) \}$$
$$\geq T \{ n \mu(\alpha x), n \mu(\beta y) \}$$
$$\geq T \{ \mu^n(\alpha x), \mu^n(\beta y) \}$$
$$\geq T \{ \mu^n(x), \mu^n(y) \}$$

$$(FGMDL2) \quad \mu^n(gx) = \mu(n(gx))$$
$$= n \mu(gx)$$
$$\geq n \mu(x)$$
$$\geq \mu^n(x)$$

$$(FGMDL3) \quad \mu^n(x \lor y) \Lambda \mu^n(x \land y) = \min \{ \mu^n((x \lor y), \mu^n(x \land y) \}$$
$$= \min \{ \mu(n(x \lor y), \mu(n(x \land y)) \}$$
$$= \min \{ n \mu_1(x \lor y), n \mu_2(x \land y) \}$$
$$\geq \min \{ n \min \{ \mu(x), \mu(y) \}, n \min \{ \mu_1(x), \mu_2(y) \} \}$$
$$\geq \min \{ n \mu(x), n \mu(y) \}$$
$$\geq T \{ \mu^n(x), \mu^n(y) \}, \text{for all } x, y \in M.$$

Therefore, $n$-dimensional fuzzy $G$-modular distributive lattice is a fuzzy $G$-modular distributive lattice.

Proposition 3.6: If $M$ and $N$ are fuzzy $G$-modular distributive lattices over a fuzzy sub-lattice $L$ then $L = M \oplus N$ is a fuzzy $G$-modular distributive lattice on $L$. Here $\oplus$ is referred as a ring sum.

Proof: Let $M$ and $N$ be two fuzzy $G$-modular distributive lattices of $\mu_1$ and $\mu_2$ respectively. Then we define the ring sum of $M.$ and $N$ as

$$(M \oplus N)(z) = \sup \{ (\mu_1 + \mu_2)(z) / x = f^{-1}(\mu_1), y = f^{-1}(\mu_2) \}$$
$$z \in M \oplus N$$

Now

$$(FGMDL1) (M \oplus N)(\alpha x + \beta y) = (\mu_1 + \mu_2)(\alpha x + \beta y)$$
$$\geq \min \{ ((\mu_1 + \mu_2)(x), (\mu_1 + \mu_2)(y)) \}$$
$$\geq T \{ \sup \mu_1(z), \sup \mu_2(z) \}$$
\[
z \in M \oplus N \quad z \in M \oplus N
\]
\[
\geq T \{(M \oplus N)(z), M \oplus N(z)\}
\]
\[
\geq T \{(M \oplus N)(x), M \oplus N(y)\}
\]

(FGMDL2) \((M \oplus N)(gz) = (\mu_1 + \mu_2)(gz) \geq (\mu_1 + \mu_2)(z) \geq \text{Sup } \{\mu_1 + \mu_2\}(z) \geq M \oplus N(z)\)

(FGMDL3)
\[(M \oplus N)(x \lor y) \land (M \oplus N)(x \land y) = \min \{ (M \oplus N)(x \lor y), (M \oplus N)(x \land y) \} = \min \{ (\mu_1 + \mu_2)(x \lor y), (\mu_1 + \mu_2)(x \land y) \} \geq \min \{ \min \{(\mu_1 + \mu_2)(x), (\mu_1 + \mu_2)(y)\}, \min \{(\mu_1 + \mu_2)(x), (\mu_1 + \mu_2)(y)\}\} \geq \min \{(\mu_1 + \mu_2)(x), (\mu_1 + \mu_2)(y)\} \geq T \{ \text{Sup } \mu_1(z), \text{Sup } \mu_2(z) \} \geq T \{(M \oplus N)(x), M \oplus N(y)\}\)

Therefore, \(M \oplus N\) is a fuzzy G-modular distributive lattices.

**Proposition 3.7:** Let \(M\) be a G-modular distributive lattices over a sub lattice \(L\) and \(M = \sum_{i=1}^{n} M_i\), where \(M_i\)'s are G sub modular distributive lattice of M. If \(V_i\) are fuzzy G-modular distributive lattice of \(M_i\) then \(V: M \rightarrow [0, 1]\) is fuzzy G-modular distributive lattice in \(L\).

**Proof:** Since \(V_i\) is a fuzzy G-modular distributive lattice on \(M_i\) for every \(x, y \in M_i\); \(g \in L\) and \(\alpha, \beta \in L\). We have

(FGMDL1) \[V(\alpha x + \beta y) = V(\Sigma(\alpha M_i + \beta M'_i)) = \Lambda(V_i(\alpha M_i + \beta M'_i)), \text{ where } i = 1, 2 \ldots n.\]
\[= V_j(\alpha M_i + \beta M'_i), \text{ for some } j\]
\[\geq T(V_j(M_i), V_j(M'_i)) \geq T(V(x), V(y))\]

(FGMDL2) \[V(gx) = V(\Sigma gM_i) = \Lambda(V_i(gM_i)), \text{ where } i = 1, 2 \ldots n.\]
\[= V_j(gM_i), \text{ for some } j\]
\[\geq V_j(M_j) \geq V(x)\]
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\[(\text{FGMDL3})\quad \forall (x \lor y) \land (x \land y) = \min \{ \forall ((x \lor y), \forall (x \land y) \} \]

\[= \min \{ \forall \sum (x \lor y)M_i, \forall \sum (x \land y) M_i' \} \]

for \(i = 1, 2, \ldots, n\)

\[= \min \{ \forall (x \lor y)M_i, \forall (x \land y) M_i' \} \]

\[\geq \min \{\min \{\forall (x), \forall (y)\} M_i, \}
\]

\[\{\min \{\forall (x), \forall (y)\} M_i'\} \]

\[\geq \min \{ \forall (x)M_i, \forall (y) M_i' \} \]

\[\geq T \{\forall_i M_i, \forall_j M_j' \}, \text{ for some } j. \]

\[\geq T \{\forall (x), \forall (y)\} \]

Therefore \(V\) is a fuzzy G-modular distributive lattice on a fuzzy G-modular distributive lattice in \(L\).

**Proposition 3.8:** Let \(\lambda\) and \(\mu\) be fuzzy G-modular distributive lattices of \(L\), then \(\lambda + \mu\) is the smallest fuzzy G-modular distributive lattice of \(L\).

**Proof:** For any \(x, y, h, f \in L\), we have

\[(\text{FGMDL1})\]

\[\min \{(\lambda + \mu)x, (\lambda + \mu)y\} = \min \{\sup \min \{(\lambda(a), \mu(b))\}, \]

\[x = a + b \]

\[\sup \min \{(\lambda(c), \mu(d))\}
\]

\[y = c + d \]

\[= \min \{\sup (\min \{(\lambda(a), \mu(b))\}, \min \{(\lambda(c), \mu(d))\}) \]

\[x = a + b \]

\[y = c + d \]

\[= \min \{\sup \min \{(\lambda(a), \mu(b), \lambda(c), \mu(d))\} \]

\[x = a + b \]

\[y = c + d \]

\[\leq \min \{\sup \min \{(\lambda(a+b-c-b), \mu(b+(c-d-c))\}) \]

\[x = a + b \]

\[y = c + d \]

\[\leq \min \{\sup \min \{(\lambda(h), \lambda(f))\} \]

\[x + y \]

\[= h + f \]

\[\leq (\lambda + \mu)(x + y) \]

Therefore \((\lambda + \mu)(x + y) \geq T \{(\lambda + \mu)(x), (\lambda + \mu)(y)\} \]

Hence (FGMDL1) is satisfied.
\[(\text{FGMDL2}) \ (\lambda + \mu) \ (gx) = \sup \min \{ (\lambda(a), \mu(b)) \} \]
\[x = a + b \]
\[= \sup \min \{ (\lambda(a), \mu(b)) \} \]
\[g_x = g_a + g_b \]
\[\geq \sup \min \{ (\lambda(A), \mu(B)) \} \]
\[X = G_A + G_A \]
\[\geq (\lambda + \mu) (x) \]

Therefore FGMDL2 is satisfied.

\[(\text{FGMDL3}) \]
\[\ (\lambda + \mu) \ (x \lor y) \land (\lambda + \mu) \ (x \land y) \]
\[= \min \{ (\lambda + \mu) (x \lor y), (\lambda + \mu) (x \land y) \} \]
\[= T \{ \max \{ (\lambda + \mu) (x), (\lambda + \mu) (y) \}, \]
\[\min \{ (\lambda + \mu) (x), (\lambda + \mu) (y) \} \} \]
\[= T \{ \max \{ \sup \min \{ (\lambda (x), \mu (x)) \}, \]
\[\sup \min \{ (\lambda (y), \mu (y)) \} \} \}
\[x = a + b \]
\[\sup \min \{ (\lambda(y), \mu(y)) \}, \]
\[y = c + d \]
\[\min \{ (\sup \min \{ \lambda (x), \mu (x) \}), \]
\[\sup \min \{ (\lambda (y), \mu(y)) \} \} \}
\[y = c + d \]
\[\geq T \{ \sup \min \{ (\lambda(y), \mu(y)) \}, \]
\[\sup \min \{ (\lambda (x), \mu (x)) \} \}
\[x = a + b \]
\[\geq T \{ \sup \min \{ (\lambda (x), \mu (x)) \}, \]
\[\sup \min \{ (\lambda(y), \mu(y)) \} \}
\[y = c + d \]
\[\geq T \{ (\lambda + \mu) (x), (\lambda + \mu) (y) \} \]

Therefore \((\lambda + \mu) (x \lor y) \land (\lambda + \mu) (x \land y) \geq T \{ (\lambda + \mu) (x), (\lambda + \mu) (y) \} \)

FGMDL3 is satisfied

Hence \(\lambda + \mu\) is a fuzzy G-modular distributive lattice on \(L\).

**Proposition 3.9:** Let \(L\) be a complete distributive lattice. If \(A\) is fuzzy G-modular distributive lattice with respect to fuzzy filter of \(L\) then its characteristic function \(A^*\) is also a fuzzy G-modular distributive lattice under fuzzy filter.

**Proof:** Since \(L\) is a complete lattice. \(A\) be a fuzzy G-modular lattice under the
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constraints of fuzzy filter. Now $A^a$ also fuzzy G-modular distributive lattice in terms of fuzzy filter constraints.

\[(\text{FGMDL1}) \quad A^a(x + y) = A((x + y) + a)\]
\[\geq T \{A(x + a), A(y + a)\}\]
\[\geq T \{A(x), A(y)\}\]

\[(\text{FGMDL2}) \quad A^a(gx) = A(gx + a)\]
\[\geq A(x + a)\]
\[\geq A^a(x)\]

\[(\text{FGMDL3}), \text{for any } x, y \in L, \text{we have}\]
\[A^a(x \lor y) \land A^a(x \land y) = \min \{A^a(x \lor y), A^a(x \land y)\}\]
\[= \min \{A(x \lor y) + a), A((x \land y) + a)\}\]
\[\geq \min \{\max \{A(x + a), A(y + a)\},\]
\[\min \{A(x + a), A(y + a)\}\}
\[(\text{since } A \text{ is fuzzy filter})\]
\[\geq \min \{A(y + a), A(x + a)\}\]
\[\geq T \{A(x + a), A(y + a)\}\]
\[\geq T \{A^a(x), A^a(y)\}\]

Therefore $A^a$ is fuzzy G-modular distributive lattice under fuzzy filter constraints.

**Proposition 3.10:** Let $\mu$ be a fuzzy subset of $X$. Then $\mu$ is fuzzy G-modular distributive lattice of $X$, if and only if Each non-empty level subset $U(\mu; t)$ of $\mu$ is fuzzy G-modular distributive lattice.

**Proof:** Assume that $\mu$ is fuzzy G-modular distributive lattice and $U(\mu; t)$ is a non-empty level subset of $X$.

\[(\text{FGMDL1}), \text{since } U(\mu; t) \text{ is non-empty level sub set. There exist } x, y \in U(\mu; t).\]
\[\mu(x + y) \geq T \{\mu(x), \mu(y)\}\]
\[\geq T \{t, t\}\]
\[\geq t, \text{thus } x + y \in U(\mu; t)\]

\[(\text{FGMDL2}), \text{Let } x \in U(\mu; t), \text{then there exist } g \in L \text{ such that}\]
\[\mu(g(x)) \geq \mu(x)\]
\[\geq t, \text{thus } g(x) \in U(\mu; t)\]

\[(\text{FGMDL3}), \text{Let } x, y \in U(\mu; t) \text{ such that}\]
\[\mu(x \lor y) = \max \{\mu(x), \mu(y)\} \text{ and}\]
\[\mu(x \land y) = \min \{\mu(x), \mu(y)\}\]
Now
\[\mu(x \lor y) \land \mu(x \land y) = \min\{\mu(x \lor y), \mu(x \land y)\}\]
\[= T\{\max\{\mu(x), \mu(y)\}\},\]
\[\{\min\{\mu(x), \mu(y)\}\}\]
\[\geq T\{\max\{t, t\}, \min\{t, t\}\}\]
\[\geq T\{t, t\}\]
\[\geq t, \text{ thus } x \lor y \text{ and } x \land y \in U(\mu; t)\]

Therefore \(U(\mu; t)\) is a fuzzy G-modular distributive lattice.

Conversely,

**(FGMDL1)** Let it possible,
\[\mu(x_0 + y_0) \leq T\{\mu(x_0), \mu(y_0)\}, \text{ for some } x_0, y_0 \in U(\mu; t).\]

Then by taking,
\[t_0 = \frac{1}{2}\{\mu(x_0 + y_0) + T\{\mu(x_0), \mu(y_0)\}\}\]

We have \(\mu(x_0 + y_0) \geq t_0\), for \(\mu(x_0) \geq t_0\) and \(\mu(y_0) \geq t_0\)
Thus \((x_0 + y_0) \notin U(\mu; t)\)

This is a contradiction and so,
\((x + y) \geq T\{\mu(x), \mu(y)\} \text{ for all } x, y \in U(\mu; t)\).

**(FGMDL2)** Let it possible, for some \(x_0 \in U(\mu; t)\).
\[\mu(g(x)) \leq \mu(x). \text{ for all } a, b \in X.\]

and then by taking
\[t_0 = \frac{1}{2}\{\mu(g(x_0)) + \mu(x_0)\}\]

We have, \(\mu(g(x_0)) + \mu(x_0) \geq t_0\), for \(\mu(x_0) \geq t_0\). Thus \(g(x_0) \notin U(\mu; t)\).
This is a contradiction. Therefore \(g(x_0) \in U(\mu; t)\).

**(FGMDL3)**
Let it possible,
\[\mu(x_0 \lor y_0) \leq \max\{\mu(x_0), \mu(y_0)\}, \text{ for some } x_0, y_0 \in U(\mu; t).\]
Then by taking
\[t_0 = \frac{1}{2}\{\mu(x_0 \lor y_0) + \min\{\mu(x_0), \mu(y_0)\}\}\]

we have \(\mu(x_0 \lor y_0) \geq t_0.\)
Thus \(x_0 \lor y_0 \notin U(\mu; t)\).

Similarly, \(\mu(x_0 \land y_0) \leq \min\{\mu(x_0), \mu(y_0)\}, \text{ for some } x_0, y_0 \in U(\mu; t)\). Then by taking
\[t_0 = \frac{1}{2}\{(x_0 \land y_0) + \max\{\mu(x_0), \mu(y_0)\}\}\]
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we have $\mu(x_0 \Lambda y_0) \geq t_0$.
Thus $x_0 \Lambda y_0 \not\in U(\mu; t)$.

Which is a contradiction to our assumption. Thus
$\mu(x \vee y) \Lambda \mu(x \Lambda y) \geq T\{\mu(x), \mu(y)\}$

Hence the proof.

Conclusions

B. Davvaz et al. investigated a new kind of fuzzy sub lattices (ideal, filter) of a lattice and I. Jahan introduce the concept of modularity of Ajmal for the lattices of fuzzy ideals of a ring. Recently M. Marudai and V. Rajendran studied the lattice structures on fuzzy G-modules. In this paper we study the characterization of fuzzy G-modular distributive lattices. One can obtain the similar results in soft modules and soft lattices by using a suitable mathematical tool.

References